Finite generation of permutation twisted cohomology

Henry Shin

Abstract

We settle Miller's Noetherianity conjecture for the full canonical effective permutation twisted cohomology ring of every finite p-group in defining characteristic. More generally, for every finite group and every finite family of actual finite-dimensional real representations, we prove that the effective section ring over any commutative Noetherian base is finite as a module over an explicit finitely generated central commutative algebra. The finite-module theorem also holds after subgroup restriction, on coordinate faces, and with bounded permutation coefficient complexes; over fields it extends to bounded complexes of finite-dimensional modules.

Over a field of positive characteristic, the section ring of the doubled regular representation gives a Noetherian bigraded model whose fully homogeneous spectrum contains the Balmer spectrum of permutation complexes as the complement of a single closed point. Suitable homogeneous localizations, followed by passage to twist degree zero, give affine Dirac charts. This provides an explicit Noetherian model for the geometric conclusion announced by Gómez and Miller.

For unit coefficients over nonzero bases, we also determine the minimum number of fully homogeneous coefficient generators of nonzero twist needed for module-finiteness. The proof combines geometric forerunners with a signed cone and induction on subgroups and coordinate faces; a uniform integral torsion bound and universal coefficients give the extension to arbitrary Noetherian bases.

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1 Introduction

We settle Miller's Noetherianity conjecture for the full canonical effective permutation twisted cohomology ring of every finite p-group in defining characteristic [20, Introduction, Open questions, Conjecture (b)]; see also [21, Conjecture 9]. We prove it by a finite-module theorem for finite families of actual finite-dimensional real representations, valid for every finite group over every commutative Noetherian base. The proof controls subgroup restrictions and faces of the effective cone simultaneously.

Permutation twisted cohomology records maps from the tensor unit into representation-sphere complexes in the permutation homotopy category. It supplies multigraded coordinate rings for the tensor-triangular study of permutation modules. These twists retain information that the ordinary graded endomorphisms of the unit do not: in this category the latter are just the coefficient ring in degree zero.

For a concrete example, let $G = C_p = \langle g \rangle$, where p is odd, and let k be a field of characteristic p. Let V be the real plane on which g acts by rotation through $2\pi/p$. Its sphere-chain model is

$$C = \left(kG \xrightarrow{g-1} kG \xrightarrow{\varepsilon} k\right) \qquad \text{in degrees } 2, 1, 0.$$

Its effective section ring is

$$R = \bigoplus_{q \geq 0} \bigoplus_{i \in \mathbb{Z}} H_i((C^{\otimes q})^G),$$

with multiplication induced by tensor product. Here q counts the sphere factors and is the *twist degree*. The twist-degree-one part has a basis a, c, b, in homological degrees $0, 1, 2$, respectively. Our theorem shows that R is finite as a module over $k[a, b]$, although $c \notin k[a, b]$, since this subalgebra has only even homological degrees. Thus the coefficient algebra controls the full ring through module-finiteness without necessarily generating it as an algebra. Example 6.4 gives the calculation and identifies the endpoint classes.

Previous work. Balmer and Gallauer introduced permutation twisted cohomology in their study of the tensor-triangular geometry of permutation modules [3]. Over a field of characteristic p, their ring, indexed by normal subgroups of index p, is finitely generated for every finite group [3, Definition 12.16 and Theorem 12.21]. Dubey and Gómez develop integral twisted cohomology and prove finite generation for the ring indexed by normal subgroups of index p in a p-group, under the hypothesis that p vanish or be a non-zero-divisor in the Noetherian base [5, Convention 8.1, Definition 8.12 and Corollary 8.13]. Their Definition 8.10 already allows abstract finite families of twists. Our theorem proves finite generation for the effective section ring of any finite family of actual finite-dimensional real representations and permits arbitrary torsion in the coefficient ring.

For general finite groups, orientability constrains the sphere sections used below [22, Introduction and Section 1.1]; doubled fixed spaces provide the integral orientations used here.

Earlier representation-graded calculations for elementary abelian groups were obtained by Holler and Kriz at the prime 2 [13, especially Theorem 5] and, for the effective cone, at odd primes [14, Theorem 3.1]. Hausmann and Schwede give minimal presentations of the effective mod-2 Bredon homology rings for elementary abelian 2-groups [10, Theorem 2.5]. Kriz studies representation localizations with constant coefficients, including a calculation for split extraspecial 2-groups [15, Section 4]. Kriz also describes the geometric fixed-point rings for elementary abelian groups in terms of reciprocal and super reciprocal planes [16, Introduction].

Miller's remixed construction uses the full canonical effective cone of representation-sphere dimension functions [20, Definition 5.1]. The underlying canonical basis is established in [19, Theorem 4.12]. For elementary abelian groups it agrees with the Balmer–Gallauer construction; for general p-groups it includes additional directions. Miller treats Noetherianity for Dedekind p-groups and for the dihedral group of order 8 in [20, Theorems 7.5–7.6]. Corollary B proves Noetherianity for this full canonical effective ring.

The geometric comparison framework is due to Miller. His Fall 2026 research statement announces joint work with Gómez obtaining comparison results without a Noetherianity hypothesis, while retaining full-ring Noetherianity as Conjecture 1.8 [23, Section 1.4, Theorems 1.7 and 1.9, and Conjecture 1.8]. Their announced manuscript, *Permutation twisted cohomology, reduced and immersed*, is listed as available upon request [9]; we cite these public announcements to describe the relationship with ongoing work. The announced results are not inputs to the proofs

below, and we do not identify their rings with ours. Our main contribution is the finite-module theorem. For finite p-groups in defining characteristic, Corollary C combines it with a transfer proof of comparison injectivity to obtain the open immersion and Dirac conclusion formulated in [20, Corollaries 6.7–6.8]. Over a field of positive characteristic, Theorem D treats arbitrary finite groups using only the doubled regular representation. Its Noetherian section ring realizes the geometric conclusion announced in the cited Theorem 1.9.

The finite-module theorem. We first state the general theorem. Its section rings are indexed by an arbitrary finite labelled family; the canonical effective ring is a particular choice of such a family, identified before Corollary B. For a finite group G, a commutative Noetherian ring Λ, and a finite ordered family $(V_b)_{b\in B}$ of actual finite-dimensional real representations, choose G-invariant inner products and finite admissible spherical polyhedral triangulations of the unit spheres $S(V_b)$, compatible with every fixed sphere $S(V_b^Q)$, $Q \le G$. Let $C_{b,\mathbb{Z}}$ be the shifted augmented chains of the chosen triangulation of $S(V_b)$, with the augmentation in degree zero; take $C_{b,\mathbb{Z}} = \mathbb{Z}$ if $V_b = 0$. Admissibility means that every setwise simplex stabilizer fixes its simplex pointwise. The hyperplane construction in Section 2 provides these simultaneous models for any finite group; Lemma 2.1 and the following join construction justify their subdivision and tensor identifications integrally. Put $C_b = C_{b,\mathbb{Z}} \otimes_{\mathbb{Z}} \Lambda$. Throughout, a permutation ΛH-module means a finite direct sum of orbit modules $\Lambda[H/J]$. Unless a statement explicitly allows arbitrary finite-dimensional coefficients over a field, coefficient complexes have such finite permutation terms. Complexes are homologically graded, and $\Lambda[i]$ is concentrated in degree i. The suspension index used in the geometric comparison is $s = -i$. Put $\mathbb{N} = \mathbb{Z}_{\ge 0}$; all zeroth and empty tensor products are Λ. Write $C_T(q) = \bigotimes_{b\in T} C_b^{\otimes q_b}$ for $T \subseteq B$ and $q \in \mathbb{N}^T$. The *face* indexed by T sets all twist coordinates outside T to zero. The vector q is the *twist*: its coordinate q_b counts copies of C_b. A nonzero twist means $q \in \mathbb{N}^T \setminus \{0\}$; individual coordinates may be zero. We call a section *fully homogeneous* if it is homogeneous in homological degree and in every formal twist coordinate. The setwise stabilizer of each doubled fixed space acts trivially on its integral orientation line. The Gómez–Miller orbit-sum construction therefore gives global maps

$$\eta_{Q,b} : \Lambda[2\dim V_b^Q] \longrightarrow C_b^{\otimes 2} \qquad (Q \le G,\ b \in B).$$

They are base changes of integral maps chosen once for all subgroups. Section products tensor representatives and regroup the sphere factors in the fixed order, with the Koszul signs. Restricted global sections act in the same way on the subgroup modules below. In $\Lambda[\eta_{Q,b}]$, the generators denote homotopy classes of these maps; their action factors through the restriction image in the section ring with coefficient complex Λ.

Theorem A. For every $H \le G$, every $T \subseteq B$, and every bounded permutation ΛH-complex D, the section module

$$\bigoplus_{q\in\mathbb{N}^T} \bigoplus_{i\in\mathbb{Z}} H_i((\mathrm{Res}^G_H C_T(q) \otimes D)^H)$$

is finite over the restriction image of $\Lambda[\eta_{Q,b} : Q \le H,\ b \in T]$. This image is a finitely generated central commutative Λ-algebra. Consequently the effective section ring of every such family is finitely generated over Λ and is left and right Noetherian. If Λ is a field, the same finite-module assertion holds for every bounded complex of finite-dimensional ΛH-modules.

Proof mechanism. For a p-group in defining characteristic, a *forerunner at H* is a global section whose image under the Brauer functor at H is a quasi-isomorphism. The proof uses

one such section at a time. After restriction to H, its cone has acyclic Brauer image at H. Elementary cancellation replaces this H-complex by a bounded permutation complex with stabilizers proper in H. A finite filtration by chain degree reduces its section module to smaller subgroups, while the face on which the selected twist coordinate is zero has one fewer label. An explicit left dg module structure on the right-multiplication cone makes these reductions compatible. On cone homology, the coefficient action factors through the restriction image of the chosen global homology classes, so every relation among them acts as zero. Descent in nonnegative twist degree then proves finite generation. For arbitrary groups and coefficient rings, doubling, Sylow transfer, a uniform group-order torsion bound, and universal coefficients complete the argument. The finiteness proof is independent of geometric comparison.

For the defining-characteristic refinement, let G be a finite p-group and let k be any field of characteristic p. Specialize the preceding sphere models to k, allowing repetitions and zero representations while retaining every formal label. Write

$$\mathcal{K}(G) = K^b(\mathrm{perm}(kG)), \qquad \mathrm{Hom}_{\mathcal{K}(G)}(k[i], C) = H_i(C^G).$$

These Hom groups are unchanged on passing to the idempotent-complete permutation category used in the literature. They are computed in the homotopy category, rather than the ordinary derived category of kG-modules. For $T \subseteq B$, with its induced order, put

$$R_T = \bigoplus_{q \in \mathbb{N}^T} \bigoplus_{i \in \mathbb{Z}} H_i(C_T(q)^G).$$

For $b \in T$, let $e_b \in \mathbb{N}^T$ have b-coordinate one and all other coordinates zero. With the multiplication convention above, for $H \leq G$ and a bounded permutation kH-complex D, define the section module

$$N_H^T(D) = \bigoplus_{q \in \mathbb{N}^T} \bigoplus_{i \in \mathbb{Z}} H_i((\mathrm{Res}^G_H C_T(q) \otimes D)^H), \qquad M_H^T = N_H^T(k). \tag{1}$$

Restriction and tensor product give these modules left R_T-actions. Every global label is retained after restriction: isomorphic restricted objects do not identify distinct twist vectors.

The geometric forerunners of Gómez–Miller [8, Construction 8.5 and Proposition 8.7(a)] are global maps

$$\theta_{J,b} : k[\dim_{\mathbb{R}} V_b^J] \longrightarrow C_b \qquad (J \leq G,\ b \in B)$$

that become quasi-isomorphisms after applying the Brauer functor at J. Section 3 reproduces their construction. Choose these maps once, and also write $\theta_{J,b}$ for their classes in R_T when $b \in T$. For $m = 1, 2$ set

$$S^{(m)}_{H,T} = k[\theta^m_{J,b} : J \leq H,\ b \in T] \subseteq R_T.$$

The coefficient actions in the following proposition are the restrictions of these global classes; injectivity of restriction is not required.

Proposition 1.1 (Defining-characteristic refinement)**.** *For every $H \leq G$, every $T \subseteq B$, and every bounded complex D of finite-dimensional kH-modules, the module $N_H^T(D)$ is finite over the image of $S^{(2)}_{H,T}$. This coefficient algebra is a finitely generated central commutative k-algebra.*

If $p = 2$, or if every V_b has even real dimension, the same assertion holds with $S^{(1)}_{H,T}$ in place of $S^{(2)}_{H,T}$. Consequently each R_T is finitely generated over k and is left and right Noetherian.

To recover the canonical effective ring, let $\mathcal{B}_G$ contain one nontrivial irreducible real representation for each distinct function $H \mapsto \dim V^H$, for $H \leq G$, and put $R_G = R_{\mathcal{B}_G}$. Equal fixed-dimension functions determine isomorphic sphere objects by [19, Theorem 4.6]. With homological degree i equal to the negative of Miller's shift index, this is precisely the ring of [20, Definition 5.1]. Nontrivial irreducible real representations of an odd-order group have even dimension, as recalled in Lemma 2.5. Thus no squares are needed in this case.

Corollary B. For every finite p-group G and every field k of characteristic p, the canonical effective permutation twisted cohomology ring R_G is finite over the central commutative algebra

$$k[\theta_{J,b} : J \leq G,\ b \in \mathcal{B}_G].$$

In particular R_G is finitely generated over k and is left and right Noetherian. At $p = 2$ it is commutative Noetherian. The same finite-module statement holds on every face and after subgroup restriction, retaining the original twist labels, with any fixed bounded finite-dimensional coefficient complex.

Put $\mathscr{K}(G) = \mathcal{K}(G)^\natural$ and $X_G = \mathrm{Spc}(\mathscr{K}(G))$, where $(-)^\natural$ denotes idempotent completion and Spc is the Balmer spectrum of prime thick tensor ideals. We use this category and spectrum notation for arbitrary finite groups as well. For the graded rings in the comparison, Spec^h denotes proper homogeneous ideals prime on homogeneous elements, with the grading indicated; $D(f)$ consists of the primes not containing the homogeneous element f. Let $\mathcal{O}_G^*$ be the sheaf obtained from the suspension-graded endomorphism rings of the tensor unit in the localized permutation categories, as defined in Section 8. The following conclusion concerns the canonical family over a field in defining characteristic.

Corollary C. Miller's comparison map

$$c_G : X_G \longrightarrow \mathrm{Spec}^h(R_G)$$

is a homeomorphism onto the open subspace

$$\bigcup_{H \leq G} D\left(\prod_{b \in \mathcal{B}_G} \theta_{H,b}\right).$$

The target has the full suspension-and-twist grading, with shift index $s = -i$. The ringed space $(X_G, \mathcal{O}_G^*)$ is a Dirac scheme in the sense of Hesselholt–Pstrągowski [11, Definition 2.29].

The proof separates the two assertions. A fixed-sphere factor makes normalizer transfer retain exactly one term after Brauer, giving injectivity without lifting individual conjugation-invariant section classes from normal subgroups. Corollary B then makes the twist-zero chart rings Noetherian. The bounded-coefficient assertion also makes their graded morphism modules finite. A local comparison argument, proved in Section 8.2, gives the chart homeomorphisms, and central localization identifies the graded sheaves.

The geometric argument also yields a model with only one twist. For an arbitrary finite group G and a field k of characteristic $p > 0$, put $U_G = 2\mathbb{R}[G]$. Write C_{U_G} for the complex C_b of the single-label family (U_G) over k, and set

$$R_G^{\mathrm{reg}} = \bigoplus_{q \geq 0,\ s \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{K}(G)}(k, C_{U_G}^{\otimes q}[s]).$$

For a p-subgroup H let $t_H : k[2[G : H]] \to C_{U_G}$ be its geometric forerunner. The twist degree q in this section ring counts copies of U_G.

Theorem D. The bigraded ring R_G^{reg} is Noetherian. Its comparison map identifies X_G homeomorphically with the open subspace

$$\bigcup_{H\leq G,\ H \text{ a } p\text{-group}} D(t_H) = \mathrm{Spec}^h_{\mathbb{Z}\times\mathbb{Z}}(R_G^{\mathrm{reg}}) \setminus \{\mathfrak{m}_+\}, \qquad \mathfrak{m}_+ = \bigoplus_{q>0,s} (R_G^{\mathrm{reg}})_{s,q}.$$

The ringed space $(X_G, \mathcal{O}_G^*)$ is a Dirac scheme. The ideal $\mathfrak{m}_+$ is the unique closed point of the ambient fully homogeneous spectrum.

The two gradings are retained in the ambient spectrum. The affine Dirac charts use suspension degree after the twist has been trivialized. The proof finds a separator with doubled actual target and embeds that target in a positive multiple of $2\mathbb{R}[G]$. A forerunner for the doubled complement preserves separation, so the construction works without a canonical irreducible basis for arbitrary finite groups. Gallauer and Miller use the same doubled regular representation and its forerunners to determine the periodic locus of derived permutation modules [7, Appendix B, especially Example B.7 and Proposition B.8].

The one-point complement realizes, for this explicit section ring, the geometric expectation following Miller's Question 1.11 [23, Section 1.4, Question 1.11 and the following paragraph]. Miller credits a verification using a different twist to ongoing work of Benson, Iyengar, Krause and Pevtsova on Schur algebras. Question 1.11 asks for explicit generators and relations for a subring recovering the spectrum; finite-module generation alone does not provide them.

Optimal coefficient bounds. A coefficient algebra sufficient for module-finiteness need not distinguish points of the Balmer spectrum under contraction. We first record the optimal coefficient counts and then describe when the coefficient algebra preserves the topological comparison.

In the notation and under the hypotheses of Theorem A, there is such a central coefficient algebra with at most $\sum_{b\in T}(\dim V_b + 1)$ homogeneous generators. For nonzero Λ, the minimum number of sections of nonzero twist, homogeneous in homological degree and every formal twist coordinate, generating a Λ-subalgebra over which the section ring with $D = \Lambda$ is finite is $\sum_{b\in T} |\Delta_{H,b}(\Lambda)|$. Here $\Delta_{H,b}(\Lambda)$ is the set of fixed dimensions at the trivial subgroup and at p-subgroups of H for primes p that are not units in Λ. The minimum is attained by a central subalgebra. These assertions are proved in Theorem 7.8 and Corollary 7.9.

For the ring R_G^{reg} of Theorem D, if $a = v_p(|G|)$, where $v_p(|G|)$ is the exponent of p in $|G|$, then R_G^{reg} is finite over a central algebra generated by $a+1$ homogeneous elements, and fewer elements homogeneous in both gradings cannot suffice. The $a+1$ dimension sums of Corollary 8.7 can be chosen as these generators, and their principal opens give an affine Dirac cover. This number is minimal among covers by fully homogeneous ambient principal opens contained in X_G. The coefficient construction in Section 7 also enters the proof of Theorem D: finiteness over the dimension sums identifies the complement of the comparison image. The lower bounds establish optimality of the coefficient count and of the specified cover.

For cyclic groups, we give such a presentation for a coefficient subring. If $G = C_{p^n}$ with $n \geq 1$, Proposition 8.12 constructs a subring isomorphic to the following Stanley–Reisner ring of a path:

$$k[X_0, \ldots, X_n]/(X_i X_j : j - i \geq 2), \qquad |X_t| = (-2p^t, 1),$$

over which R_G^{reg} is finite. Its fully homogeneous spectrum, with its unique closed point removed, is X_G. This answers the topological subring question for cyclic p-groups, with the optimal $n+1$

homogeneous coefficient generators. The cyclic path topology is due to Balmer–Gallauer [3, Proposition 8.3]; our proof gives the displayed subring and verifies its presentation by explicit chain boundaries and cyclic periodicity. We do not identify its sheaf with the full Dirac sheaf or claim a general full-ring presentation. Moreover, cyclicity is exactly the condition under which an optimal homogeneous coefficient algebra can still separate the points of X_G among finite p-groups: every optimal such algebra for a noncyclic p-group identifies distinct maximal-subgroup closed points (Corollary 8.14).

Over finite fields, Proposition 8.9 gives, for every finite group, a lower bound on the number of fully homogeneous sections of nonzero twist generating a coefficient algebra over which R_G^{reg} is finite and whose contraction separates the closed subgroup points. The bound permits unequal twists. For C_p^2 it is at least four over $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$, whereas module-finiteness alone requires three.

Finite presentation of every section ring with coefficient complex equal to the tensor unit follows from the central finite-module theorem (Corollary 7.6). Explicit relations and small topology-preserving subrings require additional work.

The nonnegative twist condition is essential: the ring allowing arbitrary integer twists need not be Noetherian, already for C_p [3, Remark 12.25]. Our result concerns the original nonnegative cone and does not assert that forerunners generate the whole ring as an algebra. Example 6.4 exhibits the distinction.

Organization of the proof. Sections 2 and 3 construct the sphere models and their forerunners. Sections 4 and 5 prove the core even-family finite-module theorem by induction on subgroups and coordinate faces. Section 6 treats parity and bounded coefficients and deduces Corollary B. Section 7 proves Theorem A for arbitrary finite groups and Noetherian bases. Theorem 7.8 and Corollary 7.9 establish the coefficient bounds stated above. Section 8 proves Corollary C, including its sheaf assertion, and proves Theorem D in Theorem 8.6 and Corollary 8.7. It then proves the separation obstruction and the cyclic subring presentation. Appendix A, independently of finite generation and the comparison transfer proof, classifies irreducible spheres whose subgroup restriction maps on sections are surjective, and proves a multiplicative norm lift for normal subgroups (Proposition A.4).

Notation guide. The main notation is collected below; changes of coefficient ring and specialized one-twist notation will be stated explicitly.

$\mathcal{K}(G), \mathscr{K}(G)$	Permutation homotopy category and its idempotent completion.
i, s	Homological section degree and suspension degree, with $s = -i$.
$q \in \mathbb{N}^T$	Twist vector counting sphere factors; $q \neq 0$ allows zero coordinates.
$R_T, M_H^T, N_H^T(D)$	Global section ring, restricted unit module, and coefficient module.
$\theta_{J,b}$	Chosen p-group forerunner into C_b; homological degree $\dim V_b^J$ and twist e_b.
$\eta_{Q,b}$	Chosen integral doubled map and its base changes; homological degree $2\dim V_b^Q$ and twist $2e_b$.
$\sigma_{b,d}$	Sum of doubled maps indexed by fixed-space dimension d; homological degree $2d$ and twist $2e_b$ (Section 7).
t_H	Forerunner into $C_{2\mathbb{R}[G]}$; suspension degree $-2[G:H]$ and twist 1.
Ambient Spec^h	Full suspension-and-twist homogeneous spectrum.
Dirac chart grading	Suspension degree after trivializing the twists.

2 Permutation complexes and sphere coefficients

Sphere chains and subdivision

The constructions in this subsection are valid for any finite acting group and any commutative coefficient ring Λ. They are defined over $\mathbb{Z}$ and preserved by base change.

For a finite simplicial complex K, let $A(K)_r$ be its augmented oriented chain module over Λ on simplices with r vertices, including $A(K)_0 = \Lambda[\varnothing]$. The boundary is the usual alternating deletion boundary, so a vertex maps to the empty simplex. Thus the unit sphere of a representation V gives homology in degree $\dim V$. This convention is isomorphic to the usual shifted augmented complex by degreewise signs if the formal shift uses the opposite differential. For a finite group P acting simplicially on K, a section of degree i is a P-equivariant chain-homotopy class of maps $\Lambda[i] \to A(K)$. Equivalently, it is an element of $H_i(A(K)^P)$: a P-invariant degree-i cycle modulo boundaries of P-invariant degree-$(i+1)$ chains. In Miller's shift convention its degree is $s = -i$.

We use finite equivariant spherical polyhedral triangulations: each simplex is the intersection of the unit sphere with a pointed simplicial cone in the representation. These can be made compatible with any finite collection of linear subspaces by taking their defining hyperplanes, adding finitely many hyperplanes whose common intersection is zero, closing the entire collection under the group, and subdividing the resulting spherical cells. Two such triangulations have a common equivariant polyhedral refinement, obtained by intersecting their cones. Barycentrically subdividing this refinement gives a common admissible triangulation.

Lemma 2.1 (Admissible subdivision)**.** *Let a finite group P act simplicially and admissibly on K. Over any commutative coefficient ring Λ, the natural subdivision map*

$$s : A(K) \longrightarrow A(\operatorname{sd} K)$$

is a P-equivariant chain homotopy equivalence. Moreover s is natural for all simplicial automorphisms of K, whether or not their action is admissible.

Proof. For an oriented simplex $[v_1, \dots, v_r]$ put

$$s[v_1, \dots, v_r] = \sum_{\pi \in S_r} \operatorname{sgn}(\pi)[\{v_{\pi(1)}\} \subset \{v_{\pi(1)}, v_{\pi(2)}\} \subset \cdots \subset \{v_{\pi(1)}, \dots, v_{\pi(r)}\}],$$

and set $s[\varnothing] = [\varnothing]$. Cancellation of the internal faces proves $\partial s = s\partial$. Relabelling vertices proves naturality, including the orientation signs on the source.

Choose a vertex $v(F)$ in each representative of the P-orbits of nonempty faces F, and extend the choices equivariantly. This is possible because the stabilizer of F fixes every vertex of F. The assignment $F \mapsto v(F)$ defines a P-simplicial map $a : \operatorname{sd} K \to K$. For each nonempty original simplex σ, both $a_* s$ and the identity send its chain into $A(\sigma)$. For a nonempty flag with largest face F, both $s a_*$ and the identity send its chain into $A(\operatorname{sd} F)$. These supporting subcomplexes are nested under taking faces and are contractible as augmented chain complexes over $\mathbb{Z}$: σ is a simplex and $\operatorname{sd} F$ is a cone on the vertex F. The stabilizer of σ fixes σ pointwise. The stabilizer of the flag preserves F, and hence fixes F and its subdivision pointwise by admissibility. Inductively fill the chain-homotopy equations in these subcomplexes on one representative of each simplex orbit, and extend equivariantly. Both composites fix the empty simplex, so the homotopy starts with zero in degree zero. This constructs the two equivariant homotopies without division or averaging. □

Here a finite subdivision K' of K triangulates the same polyhedron, with every old simplex a union of new simplices. An equivariant subdivision has the same underlying group action. Suppose K is P-admissible. Subdivision sends an oriented old simplex to the sum of its coherently oriented subdividing top-dimensional simplices, and sends the empty simplex to itself. Cancellation on interior faces makes this a P-equivariant chain map $s : A(K) \to A(K')$.

Each new vertex has a unique smallest containing old face. Its stabilizer preserves that face and fixes it pointwise, so a choice of an old vertex in it extends equivariantly. These choices define a simplicial map $a : K' \to K$. For a nonempty old simplex σ, the maps a_*s and the identity are supported in $A(\sigma)$. For a new simplex $\tau \neq \varnothing$, let F be its smallest containing old face; the maps sa_* and the identity are supported in the chains of the subdivision of F. The supports are nested under taking faces, and the relevant simplex stabilizers fix them pointwise by admissibility of K. Their augmented integral chain complexes are bounded, free and acyclic, hence contractible. On the empty simplex the maps agree, so the homotopies start with zero there. The same orbitwise filling construction gives both equivariant homotopies, integrally and after arbitrary base change. Thus the subdivision equivalence also applies to the common refinements above.

Write C_V for these shifted augmented sphere chains with coefficients in Λ. Consequently admissible spherical polyhedral models for a fixed representation have the same permutation homotopy type. Also

$$C_{V\oplus W} \simeq C_V \otimes C_W$$

equivariantly: use the join of the two sphere models, which is admissible when the group preserves each factor, and a common admissible refinement. Under the chain convention above, concatenation of the oriented vertex lists gives an exact chain identification $A(K * L) = A(K) \otimes A(L)$. Its interchange sign is the Koszul sign, because the degree of a face is its number of vertices.

Brauer detection and sphere symmetry

From this point through the end of Section 6, let G be a finite p-group and let k be a field of characteristic p, unless a different acting group is explicitly specified. A permutation module means a finite direct sum of modules $k[G/P]$. The permutation complexes used in this subsection are bounded and homologically graded, with the sphere-chain convention above. Write $I_G = \ker(kG \xrightarrow{\epsilon} k)$ for the augmentation ideal and $M_G = M/I_GM$ for the coinvariant module of a kG-module M.

Lemma 2.2 (The augmentation ideal)**.** *The augmentation ideal $I_G \subset kG$ is nilpotent. Consequently, a finite-dimensional kG-module M with $M_G = 0$ is zero.*

Proof. Induct on $|G|$. For $G \neq 1$, choose a central element z of order p. The ideal $J = (z-1)kG$ has $J^p = 0$, and $kG/J \cong k[G/\langle z\rangle]$. By induction, $I_G^n \subseteq J$ for some n, so $I_G^{np} = 0$. If $M_G = M/I_GM = 0$, iteration gives $M = I_G^{np}M = 0$. □

For a subgroup $P \leq G$, define the Brauer quotient

$$M(P) = M^P / \sum_{Q<P} \mathrm{Tr}_Q^P(M^Q), \qquad \mathrm{Tr}_Q^P(m) = \sum_{g\in P/Q} gm.$$

It is a module over $k[N_G(P)/P]$. This is an additive functor on module maps, hence acts termwise on complexes and preserves chain homotopies. It need not be exact on arbitrary

module sequences; here we apply it termwise to permutation complexes. For a finite G-set X there is a natural identification

$$k[X](P) = k[X^P]. \tag{2}$$

Indeed, the invariant vectors have a basis of P-orbit sums. Every non-singleton orbit sum is a trace from its proper stabilizer, whereas every trace from a proper subgroup has zero coefficient at a P-fixed basis element, since its index is divisible by p. For a map between permutation modules, its Brauer map is therefore obtained by retaining the coefficients between fixed basis elements.

The fixed-basis description also gives natural isomorphisms

$$(M \otimes N)(P) \cong M(P) \otimes N(P), \qquad M^*(P) \cong M(P)^*$$

for permutation modules: use $(X \times Y)^P = X^P \times Y^P$ and the dual permutation basis. These identifications preserve evaluation, the tensor symmetry, and the differentials on complexes.

Lemma 2.3 (Joint Brauer detection)**.** *A bounded complex D of permutation kG-modules is contractible if and only if $D(P)$ is acyclic for every $P \leq G$.*

Proof. The forward implication follows because Brauer preserves homotopies. For the converse, induct on the total number of transitive orbit summands in the terms of D. If this number is positive, choose an occurring stabilizer P of maximal order, replacing conjugate orbit summands by copies of $k[G/P]$ when necessary. Put $W = N_G(P)/P$. Only the summands isomorphic to $k[G/P]$ survive Brauer at P: an orbit G/Q has a P-fixed coset only if P is conjugate to a subgroup of Q, which by maximality forces equality of orders and conjugacy. Moreover, $(G/P)^P = N_G(P)/P$. Thus $D(P)$ is a nonzero bounded acyclic complex of finite free kW-modules.

At its lowest nonzero degree the incoming differential is surjective. Reduction modulo the augmentation ideal of kW remains surjective, so some matrix entry of this differential has nonzero augmentation. Let $f : k[G/P] \to k[G/P]$ be the corresponding component of the original differential. We claim that f is invertible.

Write $[P]$ for the basis coset P. Express the P-invariant vector $f([P])$ as

$$f([P]) = \sum_O a_O \sum_{x \in O} x,$$

where O runs over the P-orbits on G/P. The scalar induced by f on the one-dimensional space $k[G/P]_G$ is

$$\lambda = \sum_O a_O |O| = \sum_{|O|=1} a_O.$$

The second equality holds in characteristic p. Its right-hand side is precisely the augmentation of the corresponding Brauer matrix entry on kW, so $\lambda \neq 0$. Right exactness of coinvariants gives $(\operatorname{coker} f)_G = 0$. By Lemma 2.2, $\operatorname{coker} f = 0$. The map f is therefore surjective, and hence invertible because its source and target have the same finite dimension.

Gaussian cancellation of an invertible differential component is valid in any additive category. If the corresponding block matrix is $\left(\begin{smallmatrix} f & u \\ v & w \end{smallmatrix}\right)$ with f invertible, changes of basis make it $\left(\begin{smallmatrix} f & 0 \\ 0 & w - vf^{-1}u \end{smallmatrix}\right)$. The relation $\partial^2 = 0$ then isolates the two-term contractible summand with differential f. This removes these two adjacent orbit summands from D. The remaining terms are sums of the

other original orbit summands, and the new complex is chain homotopy equivalent to D. All its Brauer complexes remain acyclic. Repeating the argument terminates, since each cancellation removes two summands, and yields the zero complex. Thus D is contractible. □

Lemma 2.4 (Sphere criterion for permutation complexes)**.** *Let C be a bounded permutation complex. If $H(C(P))$ is one-dimensional for every $P \leq G$, concentrated in one degree depending on P, then C is tensor-invertible in the permutation homotopy category.*

Proof. Use the usual homological dual C^* and evaluation

$$\epsilon : C^* \otimes C \longrightarrow k.$$

All terms are permutation modules. Brauer commutes with duals and tensor products, and carries this map to the evaluation for $C(P)$. Over the field k, its induced map on homology is the perfect pairing of a one-dimensional space with its dual, hence is an isomorphism. Thus every Brauer complex of cone(ϵ) is acyclic. Lemma 2.3 makes this cone contractible. Evaluation is a chain homotopy equivalence and C^* is a tensor inverse of C. □

For an admissibly triangulated representation sphere, Brauer retains the fixed simplices. The shifted augmented chain complex C_V of the unit sphere consequently satisfies

$$H(C_V(P)) \cong k[\dim_{\mathbb{R}} V^P].$$

For $V^P = 0$ the fixed unit sphere is empty and the augmentation term gives $k[0]$. For $V = 0$ take $C_V = k$. Thus the preceding lemma proves invertibility of every sphere complex used here. This gives a self-contained proof of the permutation-complex case of the criterion for endotrivial complexes in [18, Theorem 3.4], credited there to Robert Boltje.

Lemma 2.5 (Parity for odd-order groups)**.** *Suppose p is odd. A real G-representation V has even dimension if and only if every fixed space V^P has even dimension. In that case V admits a G-invariant complex structure.*

Proof. For a nontrivial irreducible complex character χ of an odd-order group, its Frobenius–Schur indicator is

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \frac{1}{|G|} \sum_{g \in G} \chi(g) = 0,$$

because squaring permutes G. Thus every nontrivial irreducible real representation is of complex type. Decompose V into its trivial real summand and such complex-type summands. If $\dim V$ is even, the trivial summand also has even dimension and can be given a complex structure. The resulting complex structure on V is G-invariant and preserves every V^P. The converse follows by taking $P = 1$. □

Lemma 2.6 (Sphere symmetry and section commutation)**.** *Let $C_b = C_{V_b}$ for a finite ordered family of actual real G-representations. For homogeneous sections x, y with respective homological degrees i, j and formal twists q, r, one has*

$$xy = (-1)^{ij + \sum_b (\dim V_b) q_b r_b} yx.$$

The same formula holds in every M_H^T, with the original formal labels retained after restriction to $H \leq G$. In particular, if $p = 2$, or if all $\dim V_b$ are even, the only commutation sign is the ordinary homological sign.

Proof. Invertibility gives $\mathrm{End}(C_b \otimes C_b) \cong \mathrm{End}(k) = k$. The self-interchange of C_b is therefore a scalar. Forgetting the group action detects the scalar on ordinary top homology, where it equals $(-1)^{(\dim V_b)^2} = (-1)^{\dim V_b}$. When two block-ordered products are compared, reversing the q_b and r_b copies of label b makes $q_b r_b$ such interchanges. Naturality of symmetry on the sources $k[i]$ and $k[j]$ contributes $(-1)^{ij}$. Multiplication is defined with the same block order on both sides, which proves the formula. Formal labels remain distinct even when their sphere objects are isomorphic. Restriction preserves the equivariant chain homotopies expressing each self-interchange as its scalar, so the same argument applies to all sections of the restricted objects, including sections that do not lift to G. □

3 Forerunners and central coefficients

We use the orbit-sum forerunners of Gómez–Miller [8, Construction 8.5 and Proposition 8.7(a)]. For a p-group over k, their orientability condition is automatic. We give the construction, the determinant argument, and the fixed-subspace calculation explicitly.

Lemma 3.1 (Geometric forerunners)**.** *Let G be a finite p-group, let k have characteristic p, and let V be an actual real G-representation. There is a finite admissible sphere-chain model C_V such that, for every subgroup $H \leq G$, there is a G-equivariant chain map*

$$\theta_{H,V} : k[\dim V^H] \longrightarrow C_V$$

whose Brauer map at H is a quasi-isomorphism. The model may be chosen once for all subgroups H.

Proof. For $V = 0$ take $C_V = k$ and the identity map. Otherwise, use the simultaneous admissible sphere model of Section 2, in which every $S(V^H)$ is a subcomplex. Take its shifted augmented chains. Orient simplex-orbit representatives and transport their orientations; admissibility makes the chain terms permutation modules.

Fix H, write $U = V^H$, and put $d = \dim U$. If $d = 0$, use the bottom Euler map $k \to C_V$. The H-fixed unit sphere is empty, so its Brauer map is the identity of $k[0]$.

Suppose $d > 0$. Let N be the setwise stabilizer of the subspace U in G, and choose a nonzero augmented top fundamental cycle z_U of the subcomplex $S(U)$. It has degree d in the shifted augmented complex. The space of such cycles is one-dimensional, because $S(U)$ has no cells above dimension $d-1$. The action of N on this space is the orientation character of U. It is trivial over k: a group of odd prime-power order has no nontrivial determinant character with values in $\{\pm 1\}$, and at $p = 2$ both signs equal one in k. Thus z_U is an N-invariant cycle. Define

$$z = \sum_{g \in G/N} g z_U.$$

It is independent of the chosen coset representatives and is a G-invariant cycle, hence defines a chain map $k[d] \to C_V$.

The Brauer map retains exactly the H-fixed simplices. Any such simplex lies in $S(V^H) = S(U)$. If $gU \neq U$, then $gU \cap U$ has dimension strictly less than d, so its unit sphere contains no simplex of dimension $d-1$. Consequently the translate gz_U contributes no degree-d simplex to the Brauer image. Exactly the coset N contributes, and its contribution is z_U. This is the top fundamental cycle of the H-fixed sphere. The resulting Brauer map is a quasi-isomorphism. □

Lemma 3.2 (Removing the top subgroup)**.** *Let H be a finite p-group and D a bounded complex of finite permutation kH-modules. If $D(H)$ is acyclic, then D is chain homotopy equivalent to a bounded permutation complex whose orbit stabilizers are all proper subgroups of H.*

Proof. Decompose each term into its transitive permutation summands. Brauer at H retains precisely the summands $k[H/H] = k$ and annihilates every proper orbit. Its differential is the matrix of the components between these trivial summands. Any nonzero entry is an invertible scalar, so the corresponding pair of trivial summands can be removed by ordinary Gaussian cancellation in the original complex. The remaining terms are still sums of the original permutation summands. Brauer preserves the cancellation and acyclicity. Repeat until no nonzero scalar entry between trivial summands remains. This terminates because each cancellation removes two summands. The remaining Brauer complex has zero differential and is acyclic, hence has no terms. Consequently no trivial summands remain. □

Lemma 3.3 (Central coefficients)**.** *Suppose $p = 2$, or every V_b has even dimension. Then*

$$S_{H,T} = S^{(1)}_{H,T} = k[\theta_{J,b} : J \leq H,\ b \in T] \subseteq R_T$$

is a finitely generated central commutative algebra, and its restriction image is central in M_H^T. It is Noetherian independently of any finiteness assertion about the section modules. For an arbitrary real family, the same holds for $S^{(2)}_{H,T}$.

Proof. For even-dimensional representations at odd primes, Lemma 2.5 makes every forerunner degree even. Lemma 2.6 then makes these classes central both globally and after restriction. In characteristic two every sign is one. For an arbitrary family, a squared forerunner has even homological degree and even twist, so the same commutation formula proves its centrality. In each case the coefficient algebra is a quotient of a polynomial ring in finitely many variables. Hilbert's basis theorem therefore applies before any finiteness assertion about R_T or M_H^T. □

Until Section 6, assume $p = 2$ or that every V_b has even dimension, and write $S_{H,T}$ for the first-power algebra. Under these hypotheses the chosen forerunners generate a central commutative algebra, as used in the induction.

4 The signed left-module cone

We construct the dg module used in the subgroup–face induction. All chain complexes are homological, with differential ∂ of degree -1. For a homogeneous element x, write $|x|$ for its degree.

Use the induced order on T. Form the differential graded G-algebra

$$\mathcal{A} = \bigotimes_{b \in T} T_k(C_b), \qquad T_k(C_b) = \bigoplus_{n \geq 0} C_b^{\otimes n}.$$

The multiplication concatenates each label block. On homogeneous block tensors $a = \bigotimes a_j$, $x = \bigotimes x_j$, it is

$$ax = (-1)^{\sum_{i<j} |a_j||x_i|} \bigotimes_i (a_i x_i). \tag{3}$$

This is strictly associative and satisfies the differential Leibniz rule. For $U \subseteq T$ and $L \leq G$, write $\mathcal{A}_L^U = (\bigotimes_{c \in U} T_k(C_c))^L$ and $\mathcal{A}_L = \mathcal{A}_L^T = \mathcal{A}^L$. Then $\mathcal{A}_H$ is a left dg $\mathcal{A}_G$-module, $H(\mathcal{A}_G) = R_T$,

and $H(\mathcal{A}_H) = M_H^T$. Each twist component is a bounded complex of finite-dimensional vector spaces. All constructions respect the twist grading.

Choose $b \in T$ and a global chain map $\theta : k[d] \to C_b$. Let e_d be the generator of $k[d]$, and also write θ for the G-invariant degree-d cycle $\theta(e_d)$. Its twist is e_b. Use the source $\mathcal{A}_H \otimes k[d]$ in this shifted twist, and define

$$t(x \otimes e_d) = x\theta, \qquad K_H = \operatorname{cone}(t).$$

Right multiplication is a degree-zero map from this source and is strictly left $\mathcal{A}_G$-linear by associativity.

As a graded vector space the cone has elements $y + s(x \otimes e_d)$, where s raises degree by one; thus $|s(x \otimes e_d)| = |x| + d + 1$. Suppress e_d in the following formulas:

$$\partial_K(y, sx) = (\partial y + x\theta, -s\partial x), \tag{4}$$

$$a \cdot (y, sx) = (ay, (-1)^{|a|} s(ax)). \tag{5}$$

The second formula is the suspended left-module action. Write $L_a(v) = a \cdot v$ for this operator. Direct substitution verifies

$$\partial_K L_a - (-1)^{|a|} L_a \partial_K = L_{\partial a}.$$

If $z = \partial h$ in $\mathcal{A}_G$, then L_h is a nullhomotopy for its action on K_H, including when h has odd degree. Consequently the action on $H(K_H)$ factors through the algebra $H(\mathcal{A}_G) = R_T$; every relation among the chosen generators of $S_{H,T}$ acts as zero. The same formulas give an action of $\mathcal{A}_H$, since θ is H-invariant. Hence a global relation that becomes a boundary only after restriction also acts nullhomotopically. We retain the $\mathcal{A}_G$-action to compare all subgroups using one ambient algebra.

Put $E = \operatorname{cone}(k[d] \xrightarrow{\theta} C_b)$, so $E = C_b \oplus sk[d]$ and $\partial_E(se_d) = \theta$. Let $K_{H,+}$ denote the sum of the cone components with $q_b > 0$. It is a left dg $\mathcal{A}_G$-submodule because all twists are nonnegative.

Lemma 4.1 (Cone identification in twists with $q_b > 0$)**.** *With the twist of E declared to be e_b, there is an isomorphism of left dg $\mathcal{A}_G$-modules*

$$(\mathcal{A} \otimes \operatorname{Res}_H E)^H \;\cong\; K_{H,+}.$$

Here $\mathcal{A}$ is restricted to H on the left, and $\mathcal{A}_G$ acts by multiplication on its first factor.

Proof. First work before taking invariants. Put $\mathcal{A}_{q_b>0} = \bigoplus_{q \in \mathbb{N}^T,\, q_b>0} C_T(q)$. Multiplication is an isomorphism of graded left $\mathcal{A}$-modules

$$\mathcal{A} \otimes C_b \longrightarrow \mathcal{A}_{q_b>0}, \qquad x \otimes c \longmapsto xc.$$

Its inverse removes the final letter of the b-block, moving that letter across all later label blocks with the Koszul sign. Every word with $q_b > 0$ has exactly one such factorization. Left linearity is simply $a(xc) = (ax)c$; thus no commutation of two arbitrary letters in the same tensor algebra is being assumed.

Extend this to the cone by the map

$$\Phi(x \otimes c) = xc, \qquad \Phi(x \otimes se_d) = (-1)^{|x|} s(x \otimes e_d). \tag{6}$$

The first formula includes the block-regrouping sign in (3). For the suspended generator,

$$\partial(x \otimes se_d) = \partial x \otimes se_d + (-1)^{|x|} x \otimes \theta.$$

Applying Φ gives $(-1)^{|x|-1} s\partial x + (-1)^{|x|} x\theta$, which is exactly $\partial_K((-1)^{|x|} sx)$. The nonsuspended formula commutes with the differential by the Leibniz rule. Likewise

$$\Phi(ax \otimes se_d) = (-1)^{|a|+|x|} s(ax) = a\Phi(x \otimes se_d),$$

using (5). Thus Φ is a chain isomorphism and left-module isomorphism. It is G-equivariant since θ is global. Restricting to H and taking invariants proves the claim. In a twist q with $q_b > 0$, this is precisely

$$(C_T(q - e_b)|_H \otimes E|_H)^H \cong (K_H)_q.$$

□

Lemma 4.2 (The face $q_b = 0$ as a quotient)**.** *Let $F = T \setminus \{b\}$. There is a short exact sequence of dg left $\mathcal{A}_G$-modules*

$$0 \longrightarrow K_{H,+} \longrightarrow K_H \longrightarrow \mathcal{A}_H^F \longrightarrow 0,$$

where the quotient action factors through the algebra projection $\mathcal{A}_G^T \to \mathcal{A}_G^F$ that sends every word with $q_b > 0$ to zero. It induces the short exact sequence of $S_{H,T}$-modules

$$0 \longrightarrow H(K_{H,+}) \longrightarrow H(K_H) \longrightarrow M_H^F \longrightarrow 0. \tag{7}$$

On the last term the coefficient action factors onto $S_{H,F}$.

Proof. At $q_b = 0$ the source of t is zero, so the cone is exactly $\mathcal{A}_H^F$. All coefficients with $q_b > 0$ act as zero in the quotient; all other coefficients act by multiplication in $\mathcal{A}_H^F$. The differential preserves twist, so the sequence splits as chain complexes by the twist grading. Its connecting homomorphisms are zero, giving (7). This splitting is not asserted to be $S_{H,T}$-linear. The summand with $q_b = 0$ need not be a submodule. □

5 Relative finite generation

For a bounded permutation H-complex D define the dg left $\mathcal{A}_G$-module and its homology module

$$\mathcal{T}_H(D) = (\mathcal{A}|_H \otimes D)^H, \qquad N_H^T(D) = H(\mathcal{T}_H(D)).$$

The coefficient algebra $S_{H,T}$ acts through $H(\mathcal{A}_G)$. For a selected label b and its H-forerunner $\theta = \theta_{H,b}$, apply Section 4 to the cone K_H of right multiplication by θ. The proof reduces its part with $q_b > 0$ to proper subgroups, identifies its $q_b = 0$ quotient with a smaller face, and lifts finite generators by descent in the selected twist coordinate.

Lemma 5.1 (Equivariant cellular finiteness)**.** *Assume M_J^T is finite over $S_{J,T}$ for every proper subgroup $J < H$. If D is H-equivariantly chain homotopy equivalent to a bounded permutation complex all of whose orbit stabilizers are proper subgroups of H, then $N_H^T(D)$ is finite over $S_{H,T}$.*

Proof. For an orbit term there is a chain isomorphism

$$\mathcal{A}^J \xrightarrow{\sim} (\mathcal{A}|_H \otimes k[H/J])^H, \qquad x \longmapsto \sum_{g \in H/J} gx \otimes gJ. \tag{8}$$

It is well defined because x is J-invariant; its inverse takes the coefficient of the coset J. For $a \in \mathcal{A}_G$ we have $g(ax) = a(gx)$, so this is an isomorphism of dg left $\mathcal{A}_G$-modules. The induced action on M_J^T is exactly global restriction, with every original twist label retained. A finite set generating it over $S_{J,T} \subseteq S_{H,T}$ also generates it over $S_{H,T}$; the latter action exists independently of the induction hypothesis.

Filter the bounded model of D by its lower brutal truncations $D_{\le n}$, retaining the original terms and differential in degrees at most n and using zero above. The successive quotient is $D_n[n]$, a finite sum of orbit terms concentrated in degree n. The resulting short exact sequences of tensor complexes are split in each degree as H-modules. Taking invariants therefore preserves their degreewise exactness. All maps and connecting homomorphisms respect the $S_{H,T}$-action. By (8), each quotient homology is finite over $S_{H,T}$. The homological shifts introduce no coefficient sign, since the coefficient generators are even at odd primes and signs vanish at two. The long exact sequences and Noetherianity of $S_{H,T}$ show successively that the total homology is finite. The number of nonzero filtration quotients is bounded by the number of degrees in which the chosen bounded model of D has a nonzero term, independently of q.

Finally the construction preserves chain homotopy equivalences in D. For an H-equivariant homotopy h of degree one, the tensor homotopy is $x \otimes z \mapsto (-1)^{|x|} x \otimes h(z)$. It is graded left $\mathcal{A}_G$-linear and remains a homotopy on invariants. Thus replacement by the proper-stabilizer model is compatible with the action, even though that replacement is only H-equivariant. □

Theorem 5.2 (Even-family finiteness)**.** *Suppose $p = 2$, or every V_b has even real dimension. For every $H \le G$ and every face $T \subseteq B$, the globally labelled section algebra M_H^T is finite over the restriction image of*

$$S_{H,T} = k[\theta_{J,b} : J \le H,\ b \in T] \subseteq R_T.$$

Proof. Fix the ambient G, its sphere models, and all its global forerunners once. Induct lexicographically on $(|H|, |T|)$: for a fixed H, assume the assertion for all proper subgroups with every face, and then induct on the number of labels in the current face. At the empty face $M_H^{\varnothing} = k$ and $S_{H,\varnothing} = k$.

For nonempty T, choose $b \in T$, put $F = T \setminus \{b\}$, $d = \dim V_b^H$, and $\theta = \theta_{H,b}$. Let

$$D = \mathrm{Res}_H^G \operatorname{cone}(k[d] \xrightarrow{\theta} C_b).$$

Its Brauer complex $D(H)$ is acyclic by the forerunner property. Lemma 3.2 replaces it by a bounded permutation model with proper stabilizers. The smaller-subgroup induction and Lemma 5.1 prove that $N_H^T(D)$ is finite over $S_{H,T}$. For $H = 1$, the same cancellation gives the zero model, so this assertion reads $N_1^T(D) = 0$ and requires no smaller subgroup.

Use the right-multiplication cone K_H of Section 4. By Lemma 4.1, $H(K_{H,+})$ is $N_H^T(D)$ with its twist shifted by e_b, hence finite over $S_{H,T}$. The smaller-face induction gives that M_H^F is finite over $S_{H,F}$. In the face quotient of Lemma 4.2, the $S_{H,T}$-action factors onto that algebra. The exact sequence (7) therefore proves that $H(K_H)$ is finite over $S_{H,T}$.

The cone long exact sequence is an exact sequence of modules over $S_{H,T}$. In particular it induces an injection

$$M_H^T / t M_H^T \hookrightarrow H(K_H). \tag{9}$$

The dg action (5) induces these $S_{H,T}$-module structures. Since $S_{H,T}$ is independently Noetherian, (9) makes M_H^T/tM_H^T finite.

On $M_H^T = H(\mathcal{A}_H)$, Lemma 3.3 identifies right multiplication by θ with left multiplication by its restricted class, since d is even at odd primes. Thus the induced map t on M_H^T is the action of an element of $S_{H,T}$.

Choose finitely many homogeneous generators of the quotient and lift them to homogeneous elements $m_1, \dots, m_r$ of M_H^T. For any homogeneous m of twist q, choose a homogeneous $S_{H,T}$-linear combination of the m_i representing its quotient class. The difference is tm' for m' of twist $q - e_b$. If $q_b = 0$ the difference is zero; otherwise the total twist of m' is strictly smaller. Induction on $\sum q_b$ therefore proves that the lifts generate all of M_H^T. This completes the subgroup and face inductions.

□

6 Coefficient complexes and arbitrary representation families

We now return to the modules $N_H^T(D)$ of (1). The coefficient complex D need not be the restriction of a G-complex. The complex $(\mathcal{A}|_H \otimes D)^H$ is a dg left $\mathcal{A}_H$-module. If a cycle in $\mathcal{A}_G$ becomes a boundary in $\mathcal{A}_H$, an H-invariant primitive acts as a nullhomotopy on this module. The action on homology therefore factors through the restriction image in M_H^T. We first allow bounded permutation coefficient complexes, then isolate the supplied-map criterion and apply it to squared sphere families. Finally we retain an arbitrary bounded finite-dimensional coefficient complex inside the subgroup–face induction.

Corollary 6.1 (Bounded permutation coefficients)**.** *Under the hypotheses of Theorem 5.2, the module $N_H^T(D)$ is finite over the restriction image of $S_{H,T}$ for every subgroup $H \leq G$, every face T, and every bounded permutation kH-complex D.*

Proof. Filter D by its lower brutal truncations, as in Lemma 5.1. Each quotient is a finite sum of shifted orbit modules $k[H/J]$, now allowing $J = H$. The equivariant isomorphism (8) identifies each quotient homology with a finite sum of shifts of M_J^T, with the actual global section action. Theorem 5.2 makes each M_J^T finite over $S_{J,T} \subseteq S_{H,T}$, so the entire quotient homology is finite over $S_{H,T}$. The filtration sequences are split degreewise as H-modules, so remain exact after tensoring with $\mathcal{A}$ and taking invariants. Their long exact sequences are $S_{H,T}$-linear. Since $S_{H,T}$ is Noetherian, induction through the finite filtration proves the claim. □

Proposition 6.2 (Algebraic input criterion)**.** *Let G be a finite p-group, let k be a field of characteristic p, and let $(P_b)_{b\in B}$ be a finite ordered family of bounded permutation kG-complexes. Suppose that, for every $J \leq G$ and $b \in B$, there is a global chain map*

$$\alpha_{J,b} : k[d_{J,b}] \longrightarrow P_b$$

whose Brauer map at J is a quasi-isomorphism. If p is odd, assume that every $d_{J,b}$ is even. For this family define $N_H^T(D)$ by (1) with C_b replaced by P_b. For every $H \leq G$, $T \subseteq B$, and bounded permutation kH-complex D, the module $N_H^T(D)$ is finite over the restriction image of the central commutative algebra $k[\alpha_{J,b} : J \leq H,\ b \in T]$.

Proof. The supplied maps give one-dimensional Brauer homology for each P_b at every subgroup. Lemma 2.4 therefore makes P_b invertible. Its self-interchange is a scalar in the permutation

homotopy category. The map $\alpha_{1,b}$ identifies ordinary homology with $k[d_{1,b}]$, so the scalar is $(-1)^{d_{1,b}} = 1$ under the stated parity conditions. Restriction preserves these self-interchange homotopies. Consequently the section rings for G and every H have only the usual homological commutation sign, by the proof of Lemma 2.6. The even-degree $\alpha_{J,b}$ (or characteristic two) give central coefficient algebras, which are Noetherian by Hilbert's basis theorem.

To apply the induction, form $\bigotimes_b T_k(P_b)$. Every construction in Section 4 is a chain-level construction for these bounded permutation complexes and their global maps. For the chosen $\alpha_{H,b}$, its restricted cone has acyclic H-Brauer complex; Lemma 3.2 removes the full stabilizer. The proper-subgroup filtration of Lemma 5.1 and the face quotient of Lemma 4.2 now apply. Thus the proof of Theorem 5.2, with C_b replaced by P_b and $\theta_{J,b}$ by $\alpha_{J,b}$, gives finiteness for $D = k$: the cone makes the quotient by $\alpha_{H,b}$ finite, and descent in nonnegative twist degree lifts its generators. Finally the finite filtration in Corollary 6.1 handles every bounded permutation coefficient complex D. The dg action of the restricted tensor algebra shows that this action factors through the actual restriction image, as at the start of this section. □

The parity hypothesis can be removed by using squares of the forerunners. The distinction concerns the coefficient algebra: for an arbitrary real family at an odd prime, sections into a single sphere factor need not be central in the ordinary sense.

Corollary 6.3 (Arbitrary real families)**.** *For any finite ordered family of actual real G-representations, every $H \leq G$, every face T, and every bounded permutation kH-complex D, the module $N_H^T(D)$ is finite over $S^{(2)}_{H,T}$. If $p = 2$ or every V_b has even dimension, it is also finite over $S^{(1)}_{H,T}$. These coefficient algebras are central and commutative in the respective cases. Each R_T is finitely generated over k and is left and right Noetherian; at $p = 2$ it is commutative.*

Proof. The first-power assertion is Corollary 6.1. For the squared assertion, take any real family and write

$$E_b = C_b^{\otimes 2}, \qquad \theta^2_{J,b} : k[2 \dim V_b^J] \longrightarrow E_b.$$

The complexes E_b are bounded permutation complexes. The Brauer functor preserves tensor products, so each $(\theta^2_{J,b})(J)$ is a quasi-isomorphism, and the degrees of all these maps are even. Proposition 6.2 therefore applies directly to (E_b) with these supplied maps. In particular, the self-interchange of E_b is the identity in the permutation homotopy category; no strict chain-level identity of the tensor flip is needed. We identify the formal twist n for this family with the original twist $2n$.

We check that this application respects the action on the original family, including its labels and multiplication. In its ordered tensor algebra

$$\mathcal{A} = \bigotimes_{b \in T} T_k(C_b)$$

the subalgebra of even twists is

$$\mathcal{A}_{\mathrm{ev}} = \bigotimes_{b \in T} T_k(C_b^{\otimes 2}) = \bigoplus_{n \in \mathbb{N}^T} C_T(2n).$$

This is an equality of differential graded G-algebras, using concatenation in each ordered block. For $\epsilon \in \{0,1\}^T$, let

$$\mathcal{A}_\epsilon = \bigoplus_{n \in \mathbb{N}^T} C_T(2n + \epsilon).$$

Multiplication gives an isomorphism of differential graded left $\mathcal{A}_{\mathrm{ev}}$-modules

$$\mathcal{A}_{\mathrm{ev}} \otimes C_T(\epsilon) \xrightarrow{\sim} \mathcal{A}_\epsilon, \qquad x \otimes z \longmapsto xz. \tag{10}$$

Indeed, its inverse removes the last ϵ_b letters of each b-block, moving them to the final tensor factor with the Koszul sign. Thus it is a graded vector-space isomorphism. The Leibniz rule makes it a chain map, and associativity gives $a(xz) = (ax)z$, so it is a G-equivariant isomorphism of left dg $\mathcal{A}_{\mathrm{ev}}$-modules. The construction preserves the order within each label block and retains all formal labels.

Restrict (10) to H, tensor with D, and take H-invariants. It identifies the residue-ϵ part of $N_H^T(D)$ with the even-family coefficient module having bounded permutation coefficient complex $\mathrm{Res}_H^G C_T(\epsilon) \otimes D$. By the preceding application of Proposition 6.2, this is finite over the algebra generated by the $\theta_{J,b}^2$. Under the inclusion $\mathcal{A}_{\mathrm{ev}} \subseteq \mathcal{A}$, its action is exactly the restricted action of $S_{H,T}^{(2)}$. There are only $2^{|T|}$ residues, so their sum $N_H^T(D)$ is finite over that algebra.

By Lemma 3.3, $S_{H,T}^{(2)}$ is a finitely generated central commutative algebra. Taking $H = G$ and $D = k$ makes R_T a finite module over its Noetherian central subalgebra $S_{G,T}^{(2)}$. Every left or right ideal is a submodule over that algebra, so ascending chains of either kind stabilize. Finite algebra generators of $S_{G,T}^{(2)}$ together with finite module generators of R_T generate R_T over k. At $p = 2$ the commutation formula makes R_T commutative. This proves the permutation-coefficient case of Proposition 1.1, the sharper p-group form of Theorem A. Choosing the canonical family gives the corresponding case of Corollary B, with first-power coefficients by parity. Proposition 6.6 below gives the stated extension to arbitrary finite-dimensional coefficients. □

Example 6.4 (Endpoint forerunners for a cyclic group). Let p be odd, let $G = C_p$, and let V be a nontrivial irreducible two-dimensional real representation. Choose a generator g whose action on V is rotation through $2\pi/p$. Its sphere complex has the model

$$C = (kG \xrightarrow{g-1} kG \xrightarrow{\varepsilon} k) \qquad \text{in degrees } 2, 1, 0.$$

Writing $N = 1 + g + \cdots + g^{p-1}$, the invariant terms in degrees two and one are both kN. Their differentials vanish. Consequently the section space of twist degree one has a basis

$$a \in H_0(C^G), \qquad c \in H_1(C^G), \qquad b \in H_2(C^G).$$

Here a is the bottom Euler section and b is the top fundamental section. They may be chosen as the forerunners at G and 1, respectively. The middle class c is nonzero and satisfies $c^2 = 0$ by graded commutativity. Since a and b have even homological degree, $c \notin k[a, b]$, although Theorem 5.2 makes the full section ring finite as a module over $k[a, b]$.

Remark 6.5. The squares in the arbitrary-family statement address an actual sign issue. At an odd prime, take two separately labelled trivial real lines. Their sphere complexes are $k[1]$ up to homotopy, and the degree-one sections x and y in their respective twists satisfy $xy = -yx \neq 0$. Thus first-power forerunners need not be central for this family. The squared coefficients have both even degree and even twist.

An arbitrary finite-dimensional coefficient complex D need not admit the permutation-term filtration used above. We therefore retain D throughout the subgroup–face induction. To identify the resulting cone map with the coefficient action, we use the equivariant self-interchange homotopies before tensoring with D.

Proposition 6.6 (Arbitrary finite-dimensional coefficients)**.** *With the hypotheses and P_b-family notation of Proposition 6.2, the same finite-module conclusion holds for every bounded complex D of finite-dimensional kH-modules. Its terms need not be permutation modules. In particular, Proposition 1.1 and Corollary B admit these coefficient complexes.*

Proof. Write $\mathcal{A} = \bigotimes_{b\in T} T_k(P_b)$ and $\mathcal{T}_H(D) = (\mathcal{A}|_H \otimes D)^H$. The dg left $\mathcal{A}_H = \mathcal{A}^H$-action shows that the homology action factors through the actual restricted coefficient algebra. We induct on $(|H|, |T|)$, proving the assertion simultaneously for every bounded finite-dimensional coefficient complex at each subgroup and face. For the empty face, $H(D^H)$ is finite-dimensional over k.

Fix a nonempty face and choose $b \in T$. Put $\alpha = \alpha_{H,b}$, $d = d_{H,b}$ and $E = \mathrm{cone}(k[d] \xrightarrow{\alpha} P_b)|_H$. The Brauer-acyclicity and cancellation argument replaces E by a bounded permutation complex with proper stabilizers. Filter this complex E, retaining D throughout. A filtration quotient is a finite sum of shifted $k[H/J] \otimes D$ with $J < H$. On an invariant sum, the isomorphism

$$(\mathcal{A}|_J \otimes \mathrm{Res}^H_J\, D)^J \xrightarrow{\sim} (\mathcal{A}|_H \otimes k[H/J] \otimes D)^H$$

is given by

$$\sum_\ell x_\ell \otimes z_\ell \longmapsto \sum_{g\in H/J} \sum_\ell gx_\ell \otimes gJ \otimes gz_\ell.$$

Its inverse takes the coefficient of the coset J, and it respects the global coefficient action. The smaller-subgroup induction therefore makes each quotient module finite. The filtration is split degreewise as an H-module filtration before taking invariants; its exact homology sequences and Noetherianity make $N_H^T(E \otimes D)$ finite. Tensoring an equivariant chain homotopy in E with D and $\mathcal{A}$ preserves equivariant chain homotopy equivalences.

Use right insertion of α on the $\mathcal{A}$ factor to define

$$t : \mathcal{T}_H(D) \otimes k[d] \longrightarrow \mathcal{T}_H(D), \qquad t((x \otimes z) \otimes e_d) = x\alpha \otimes z,$$

with its twist shifted by e_b. The usual interchange would contribute $(-1)^{d|z|}$, which is one because d is even at odd characteristic and signs vanish in characteristic two. Thus t is a chain map and is strictly left $\mathcal{A}_G$-linear. Its cone is a dg left $\mathcal{A}_G$-module. The part in twists with $q_b > 0$ is $(\mathcal{A} \otimes E \otimes D)^H$: the nonsuspended map is $x \otimes c \otimes z \mapsto xc \otimes z$, and the suspended map is

$$x \otimes se_d \otimes z \longmapsto (-1)^{|x|} s((x \otimes z) \otimes e_d).$$

The calculation in Lemma 4.1 applies with D retained. On the suspended term its differential contributes $(-1)^{d+1} = -1$, exactly the cone sign, while the source interchange contributes $(-1)^{d|z|} = 1$ under the stated parity convention. At $q_b = 0$ the cone is the smaller-face module with coefficient D. The sequence split by the twist grading, together with the smaller-face induction, therefore makes the full cone homology finite.

It remains to identify $H(t)$ with the coefficient action of α. The proof of Proposition 6.2 makes the self-interchange of each P_b equivariantly chain homotopic to the identity. For a fixed twist q, put $X = \bigotimes_c P_c^{\otimes q_c}$. Then naturality of symmetry moves $\alpha : k[d] \to P_b$ through X. After regrouping the ordered label blocks, the discrepancy between left and right insertion consists only of q_b self-interchanges of P_b. Their composite is homotopic to the identity. Since $(-1)^{d|x|} = 1$ under the same parity convention, left and right insertion define equivariantly homotopic degree-d maps on X. Tensor the homotopy with D, then take H-invariants. Consequently $H(t)$ is exactly multiplication by the restricted class α on $N_H^T(D)$.

The cone sequence now embeds $N_H^T(D)/\alpha N_H^T(D)$ into the finite cone homology. This quotient is finite over the Noetherian coefficient algebra. Lift finitely many homogeneous generators. Subtracting a lift leaves $\alpha m'$ with b-twist smaller by one, so descent in the nonnegative twist degree proves that the lifts generate $N_H^T(D)$. This completes the simultaneous induction. For arbitrary real families, apply this result to the squared family and use the residue decomposition (10); its new coefficient complex $C_T(\epsilon)|_H \otimes D$ is still bounded and finite-dimensional. Together with the first-power case above, this completes Proposition 1.1 and Corollary B for all their stated coefficient complexes. □

7 Finite groups and Noetherian coefficient rings

We now allow an arbitrary finite group G and a commutative Noetherian coefficient ring Λ. This section proves the general form of Theorem A. The geometric inputs are integral representation spheres and doubled forerunners, already constructed in [8, Theorem 5.6, Construction 8.5 and Proposition 8.7(a)]. We give the needed arguments, including the passage from fields to $\mathbb{Z}$ and then to Λ.

Choose integral admissible sphere complexes $C_{b,\mathbb{Z}}$ using the simultaneous fixed-subspace triangulations of Section 2. For any commutative coefficient ring A, put $C_{b,A} = C_{b,\mathbb{Z}} \otimes_{\mathbb{Z}} A$ and $E_{b,A} = C_{b,A}^{\otimes 2}$. For $n \in \mathbb{N}^T$ write

$$E_{T,A}(n) = \bigotimes_{b \in T} E_{b,A}^{\otimes n_b}.$$

For $H \leq G$, $T \subseteq B$, and a bounded AH-complex D, define

$$N_{H,A}^{T,\mathrm{ev}}(D) = \bigoplus_{n \in \mathbb{N}^T,\, i \in \mathbb{Z}} H_i((\mathrm{Res}_H^G E_{T,A}(n) \otimes_A D)^H), \qquad M_{H,A}^{T,\mathrm{ev}} = N_{H,A}^{T,\mathrm{ev}}(A).$$

The vector n records exponents in the doubled family; its original formal twist is $2n$. We use the notation R_T, $N_H^T(D)$ and M_H^T of Section 1 over Λ, with Λ in place of k and $M_H^T = N_H^T(\Lambda)$. The join of two copies of the chosen triangulation is G-admissible, since G preserves each of its two factors. Its shifted augmented chain complex is exactly

$$E_{b,\mathbb{Z}} = C_{b,\mathbb{Z}}^{\otimes 2}.$$

The fixed subspace of the doubled representation at $Q \leq G$ is $V_b^Q \oplus V_b^Q$. Its setwise stabilizer acts on its orientation line by the square of a determinant, hence trivially over $\mathbb{Z}$. For each nonzero doubled fixed subspace $V_b^Q \oplus V_b^Q$, choose a primitive integral fundamental cycle of its unit sphere in shifted degree $2 \dim V_b^Q$, and transport the choices within each G-orbit. For the zero fixed subspace use the bottom Euler map. The orbit-sum construction then gives integral global maps

$$\eta_{Q,b} : \mathbb{Z}[2 \dim V_b^Q] \longrightarrow E_{b,\mathbb{Z}} \qquad (Q \leq G). \tag{11}$$

At $Q = 1$ choose the primitive top fundamental cycle, with compatible tensor orientations. After reduction modulo p, the map $\eta_{Q,b}(Q)$ is a quasi-isomorphism for every p-subgroup Q: the fixed-simplex argument of Lemma 3.1 is unchanged. These maps need not be squares of global maps into $C_{b,\mathbb{Z}}$.

Lemma 7.1 (Integral sphere symmetry)**.** *Every $C_{b,\mathbb{Z}}$ is invertible in $K^b(\mathrm{perm}(\mathbb{Z}G))$. Its self-interchange is multiplication by $(-1)^{\dim V_b}$. Both assertions persist after base change to any commutative ring.*

Proof. Put $C = C_{b,\mathbb{Z}}$ and use its homological dual $C^* = \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$. Let Q be the cone of the evaluation $C^* \otimes C \to \mathbb{Z}$. It is a bounded integral permutation complex. For every prime p, restrict $Q \otimes \mathbb{F}_p$ to a Sylow p-subgroup P of G. Every Brauer complex of this restriction is acyclic, because fixed spheres have one-dimensional homology and evaluation is its perfect pairing. Lemma 2.3 gives a P-equivariant contraction. Averaging this contraction over G/P and dividing by $[G : P]$ gives a G-equivariant contraction over $\mathbb{F}_p$.

The complex $\mathrm{Hom}_{\mathbb{Z}G}(Q, Q)$ is bounded and finite free over $\mathbb{Z}$. Its formation commutes with reduction modulo p: for finite G-sets X, Y, the matrices which are 1 on one diagonal G-orbit of $Y \times X$ and 0 elsewhere form a $\mathbb{Z}$-basis of $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}[X], \mathbb{Z}[Y])$. Their reductions form the corresponding basis over $\mathbb{F}_p$. Universal coefficients therefore give

$$H_0 \mathrm{Hom}_{\mathbb{Z}G}(Q, Q)/p \hookrightarrow H_0 \mathrm{Hom}_{\mathbb{F}_pG}(Q \otimes \mathbb{F}_p, Q \otimes \mathbb{F}_p) = 0.$$

A finitely generated abelian group with zero quotient modulo every prime is zero. Thus the identity of Q is nullhomotopic, and evaluation is a homotopy equivalence integrally.

Invertibility identifies the endomorphisms of $C_{b,\mathbb{Z}}^{\otimes 2}$ with $\mathbb{Z}$. Ordinary top homology detects the interchange scalar, which is $(-1)^{\dim V_b}$. The equivalences and homotopies base change to any coefficient ring. □

Consequently the section commutation formula of Lemma 2.6 holds integrally and over Λ. Each $\eta_{Q,b}$ has even degree and twist $2e_b$, so these classes are central. For $H \leq G$ and $T \subseteq B$, put

$$P_{H,T,\mathbb{Z}} = \mathbb{Z}[x_{Q,b} : Q \leq H,\ b \in T], \qquad x_{Q,b} \longmapsto \eta_{Q,b}.$$

We work first over this formal polynomial algebra, without assuming flatness of its image. The actions on section modules factor through the actual image of the restricted global classes. For any coefficient ring A as above, set

$$P_{H,T,A} = A[x_{Q,b} : Q \leq H,\ b \in T] = P_{H,T,\mathbb{Z}} \otimes_{\mathbb{Z}} A.$$

Its action sends $x_{Q,b}$ to the class of $\mathrm{res}^G_H(\eta_{Q,b} \otimes_{\mathbb{Z}} A)$. This operator increases the doubled-family exponent by e_b and the original twist by $2e_b$.

Lemma 7.2 (The field step)**.** *For every $H \leq G$, every $T \subseteq B$, and every bounded permutation $\mathbb{F}_pH$-complex D, the module $N^{T,\mathrm{ev}}_{H,\mathbb{F}_p}(D)$ is finite over $P_{H,T,\mathbb{F}_p}$.*

Proof. Let P be a Sylow p-subgroup of H. Restrict the doubled family and all global maps $\eta_{Q,b}$ for $Q \leq P$ to P. Their degrees are even and their Q-Brauer maps are quasi-isomorphisms. Proposition 6.2 makes the module at P finite over these classes, hence over the larger acting algebra $P_{H,T,\mathbb{F}_p}$. For every complex X of $\mathbb{F}_pH$-modules, the chain maps

$$X^H \longrightarrow X^P \xrightarrow{[H:P]^{-1}\,\mathrm{Tr}^H_P} X^H$$

have composite the identity. Apply them for each doubled-family exponent n to $X = E_{T,\mathbb{F}_p}(n)|_H \otimes D$. They commute with the global coefficient operators: for an H-invariant coefficient cycle a, $\mathrm{Tr}^H_P(ax) = a\,\mathrm{Tr}^H_P(x)$. The section module at H is a submodule of the finite section module at P over a Noetherian polynomial algebra, and is therefore finite. □

Lemma 7.3 (The integral step)**.** *For every $H \leq G$ and $T \subseteq B$, the module $M^{T,\mathrm{ev}}_{H,\mathbb{Z}}$ is finite over $P_{H,T,\mathbb{Z}}$.*

Proof. Fix H, T, write $P_{H,T,\mathbb{Z}} = P_0$, and put $M = M_{H,\mathbb{Z}}^{T,\mathrm{ev}}$. Let N be the corresponding module after forgetting the group action. Primitive top fundamental classes identify N with a polynomial algebra over $\mathbb{Z}$, with one generator for each label. They are the actions of $x_{1,b}$, so N is finite over P_0. The image I of restriction $M \to N$ is a submodule of N, hence finite. Its kernel K is annihilated by $|H|$, because transfer back from 1 to H composed with restriction is multiplication by $|H|$.

Invariant chains are free abelian with orbit-sum bases, and their formation commutes with base change. For every prime $p \mid |H|$, universal coefficients thus identify M/pM with a submodule of the mod-p even-family module. Lemma 7.2 and Noetherianity of P_0/pP_0 imply that M/pM is finite. Since N is free abelian, its subgroup I is torsion-free. Reducing $0 \to K \to M \to I \to 0$ modulo p therefore gives

$$0 \longrightarrow K/pK \longrightarrow M/pM \longrightarrow I/pI \longrightarrow 0.$$

The kernel K/pK is consequently finite over the Noetherian algebra P_0/pP_0.

Write $|H| = \prod p^{a_p}$. Bézout's identity gives idempotents that split K into its finitely many primary summands K_p, each killed by p^{a_p}. The quotient K_p/pK_p is finite. Lift a finite set of its generators and let L_p be the submodule they generate. Then $K_p = L_p + pK_p$, and finite iteration gives $K_p = L_p + p^{a_p}K_p = L_p$. Thus K, and then M, are finite. If $H = 1$, restriction already identifies $M = N$ and no torsion argument is needed. The bound $|H|K = 0$ is uniform in all twists and homological degrees. □

The integral argument uses three inputs: finite generation after forgetting the group action, finite generation modulo the relevant primes, and a uniform group-order bound on the restriction kernel. The same argument works over a localization of $\mathbb{Z}$; we use this variant in Theorem 7.8.

Proof of the finite-module and Noetherian assertions of Theorem A. First use even twists and unit coefficients. Let M be the finite integral module of Lemma 7.3. The invariant chain complex over Λ is the integral invariant complex tensored with Λ. For a doubled-family exponent n, write $M_{i,n} = H_i((E_{T,\mathbb{Z}}(n)|_H)^H)$. Universal coefficients for each doubled-family exponent n give the natural exact sequence

$$0 \longrightarrow M_{i,n} \otimes_{\mathbb{Z}} \Lambda \longrightarrow H_i((E_{T,\Lambda}(n)|_H)^H) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(M_{i-1,n}, \Lambda) \longrightarrow 0. \tag{12}$$

The direct sums of the outer terms are $M \otimes_{\mathbb{Z}} \Lambda$ and the homological suspension of $\operatorname{Tor}_1^{\mathbb{Z}}(M, \Lambda)$. Both are finite over $P_{H,T,\Lambda} = \Lambda[x_{Q,b}]$. For the Tor assertion, resolve M by a free $P_{H,T,\mathbb{Z}}$-resolution whose terms have finite rank. Such a resolution exists by Noetherianity; it need not have finite length, and its terms are $\mathbb{Z}$-flat. After tensoring with Λ, its first homology computes the stated Tor group. Kernels and images in its finite free $P_{H,T,\Lambda}$-terms are finite, because this polynomial ring is Noetherian. Here the resolution is taken over the formal polynomial algebra, not its possibly nonflat actual image.

Naturality of universal coefficients for each even-degree chain operator gives the required polynomial action on the exact sequence. By Lemma 7.1 and the commutation formula above, the commutators of the integral coefficient cycles are boundaries of G-invariant integral chains. Left multiplication by these primitives gives homotopies between the composite operators. These homotopies persist after restriction, arbitrary base change, and taking invariants. Naturality and homotopy invariance identify the induced tensor and Tor actions. A simultaneous strict polynomial action on the chain complex is not required. Each twist is a bounded complex of free

abelian groups, and taking the direct sum of these natural exact sequences is exact; no global bound on the amplitudes of all twists is used. Therefore the even-family unit module over Λ is finite. No flatness assumption on Λ or on the actual coefficient image is involved.

This holds for every subgroup with the same original global labels and coefficient classes. Filter a bounded permutation ΛH-complex D by its lower brutal truncations. The orbit-module isomorphism (8) identifies each successive module with a finite sum of shifted subgroup modules. The filtration is split degreewise before taking invariants, so its long exact homology sequences are exact and coefficient-linear. This proves finiteness for every bounded permutation coefficient complex D.

Finally apply the dg module isomorphism (10) over Λ, restrict to H, and tensor with D. For each $\epsilon \in \{0,1\}^T$, it identifies the summand of $N_H^T(D)$ with twists $q \equiv \epsilon \pmod 2$ with

$$N_{H,\Lambda}^{T,\mathrm{ev}}(\mathrm{Res}_H^G C_T(\epsilon) \otimes_\Lambda D),$$

where exponent n is regraded as original twist $2n + \epsilon$. There are finitely many such summands. Their sum is therefore finite over the polynomial algebra, and hence over its actual image. The image is central by Lemma 7.1, and is a finitely generated commutative Λ-algebra. With $H = G$ and $D = \Lambda$, every left or right ideal of the section ring is a submodule over this Noetherian central algebra. Both ascending chain conditions follow, as does finite generation as a Λ-algebra. □

The arbitrary-coefficient field assertion is proved in Corollary 7.4 below. The coefficient bound and its optimality are proved in Theorem 7.8 and Corollary 7.9, using only the finite-module argument just established.

Corollary 7.4 (Finite groups over fields)**.** *If $\Lambda = k$ is a field, the finite-module assertion of Theorem A holds for every bounded complex D of finite-dimensional kH-modules, without a permutation-module hypothesis on its terms.*

Proof. In positive characteristic apply Proposition 6.6 to the doubled family on a Sylow subgroup. The coefficient-linear restriction and averaged transfer splitting in Lemma 7.2 works for arbitrary coefficient modules and proves the even-family claim at H. In characteristic zero, taking invariants is exact. Each primitive top section $\eta_{1,b}$ is a quasi-isomorphism $k[2\dim V_b] \to E_{b,k}$; after tensoring with D and taking invariants it identifies all successive even twists. Hence the even-family module is generated by the finite-dimensional space $H(D^H)$ over the top-section polynomial algebra. The finite parity decomposition then proves the assertion in both cases. □

Remark 7.5 (Retract coefficients over a Noetherian base)*.* The finite-module assertion of Theorem A also holds when the coefficient object is a retract of a bounded finite permutation complex in the homotopy category. The section module preserves that retract, so it is a submodule of a finite module over the same Noetherian central algebra. This includes every bounded complex whose terms are direct summands of finite permutation modules: choose a complement in a permutation module in each degree, put zero differentials on those complements, and extend the original differential by zero to obtain a bounded permutation complex containing the given complex as a direct summand.

Finite presentation and growth

Corollary 7.6 (Existence of finite presentations)**.** *Every section ring in Theorem A with coefficient complex Λ is finitely presented as an associative algebra over Λ. Its generators and relations may be chosen homogeneous in the full grading.*

Proof. Theorem A makes the section ring R finite over a finitely generated commutative central algebra S over the Noetherian coefficient ring Λ. Thus S is Noetherian and finitely presented over Λ. Choose homogeneous S-module generators $r_0 = 1, r_1, \dots, r_t$ of R. The kernel of the resulting finite free graded S-module surjection onto R has finitely many homogeneous generators. Choose also an S-linear expression for each product $r_i r_j$. Starting with a finite presentation of S, adjoin symbols for the r_i, impose $r_0 = 1$, centrality of the S-generators, and those module and multiplication relations. Every word reduces to an S-linear combination of the r_i, and the chosen module relations kill precisely the combinations mapping to zero. This is a finite homogeneous presentation of R. □

Corollary 7.7 (Rational series and parity growth)**.** *Over a field k, every module in Theorem A has rational multigraded Hilbert series. For example, with $d_{Q,b} = 2\dim V_b^Q$, it has the form*

$$\sum_{q,i} \dim_k N_H^T(D)_{i,q}\, z^i t^q = \frac{P(z,t)}{\prod_{Q\le H,\, b\in T}(1 - z^{d_{Q,b}} t_b^2)},$$

where P is a Laurent polynomial in z and a polynomial in the twist variables. The equality is interpreted as a formal power series in the twist variables with Laurent-polynomial coefficients in z. For each $\epsilon \in \{0,1\}^T$, the function $n \mapsto \sum_i \dim_k N_H^T(D)_{i,2n+\epsilon}$ agrees with a polynomial in the coordinates of n once all those coordinates are sufficiently large. In one twist variable, the total dimension is therefore eventually a quasipolynomial of period dividing two.

Proof. Apply the multigraded Hilbert–Serre formula to the finite module over the displayed polynomial algebra. Each fixed twist has only finitely many nonzero homological degrees, since its coefficient complex is bounded. Thus setting $z = 1$ is legitimate coefficientwise. The denominator then factors as a product of powers of $1 - t_b^2$; coefficient extraction gives a polynomial on each parity class once all coordinates are sufficiently large. Zero or repeated coefficient classes may make the displayed denominator nonminimal. □

Dimension-indexed coefficients

Let $\mathcal{P}$ consist of the trivial subgroup and all prime-power subgroups of G. For each label b and each G-orbit $[U]$ of actual fixed subspaces $U = V_b^P$, $P \in \mathcal{P}$, choose the integral doubled orbit-sum map $\eta_{[U],b}$. At $U = V_b$ choose the primitive top fundamental class.

Define, once over $\mathbb{Z}$,

$$\sigma_{b,d} = \sum_{\substack{[U]\in G\backslash\{V_b^P : P\in\mathcal{P}\} \\ \dim U = d}} \eta_{[U],b}.$$

All summands have homological degree $2d$ and twist $2e_b$. For a nonzero commutative Noetherian base Λ, let Π_Λ consist of the primes dividing $|G|$ which are not units in Λ, and put

$$\Delta_{H,b}(\Lambda) = \{\dim V_b^P : P \le H,\ P = 1 \text{ or } P \text{ a } p\text{-group for } p \in \Pi_\Lambda\}.$$

Theorem 7.8 (Coefficients indexed by fixed-space dimension)**.** *The central algebra*

$$S_{H,T,\Delta} = \Lambda[\sigma_{b,d} : b \in T,\ d \in \Delta_{H,b}(\Lambda)]$$

acts through its central restriction image in M_H^T and makes $N_H^T(D)$ in Theorem A a finite module. The displayed family has at most $\sum_{b\in T}(\dim V_b + 1)$ generators, independently of the number of subgroups of G.

Proof. Fix a p-subgroup P, and put $d = \dim V_b^P$. Apply Brauer at P in characteristic p to $\sigma_{b,d}$. A summand from another fixed-space orbit is supported on translates of a doubled d-dimensional subspace distinct from $V_b^P \oplus V_b^P$. Their intersections have dimension less than $2d$, so none contributes a P-fixed simplex in degree $2d$. The unique orbit containing V_b^P contributes exactly its fundamental cycle, by the setwise-stabilizer orbit sum. Thus $\sigma_{b,d}(P)$ is a quasi-isomorphism whenever $\dim V_b^P = d$. The zero fixed subspace, whose unit sphere is empty, has a single orbit and the bottom Euler map. At $P = 1$, the unique top-dimensional subspace is V_b, so $\sigma_{b,\dim V_b}$ is primitive.

In characteristic $p > 0$, apply Proposition 6.2 on a Sylow p-subgroup of H to the doubled family and the maps $\sigma_{b,\dim V_b^P}$ indexed by its subgroups P. These maps have even degrees and the required Brauer quasi-isomorphisms; the same map may serve several subgroups. The coefficient-linear restriction and transfer argument in the proof of Lemma 7.2 gives finite generation over the dimension-sum algebra. For arbitrary finite-dimensional coefficients in characteristic p, apply Proposition 6.6 to these maps and use the same restriction and transfer. In characteristic zero, the primitive top sections and exactness of invariants give the result as in the proof of Corollary 7.4.

For the general base, put $A = \mathbb{Z}[1/m]$, where m is the product of the primes dividing $|G|$ which are units in Λ, with $m = 1$ for the empty product. Then Λ is an A-algebra. Repeat the integral argument over A. The restriction kernel is killed by $|H|$; its primary pieces at inverted primes vanish, and the remaining pieces lift from the mod-p finite modules by the same finite iteration. The restriction image is finite because the primitive top classes are retained. Universal coefficients over the PID A, followed by the coefficient filtration and parity-residue decomposition, give the conclusion over Λ. Every acting map has even homological degree and even twist, hence is central. Finally $\Delta_{J,b}(\Lambda) \subseteq \Delta_{H,b}(\Lambda)$ for $J \leq H$; all maps were fixed globally, so the subgroup induction uses the required literal inclusions of coefficient algebras. □

Corollary 7.9 (Optimal homogeneous coefficient count for a finite family)**.** *Let Λ be a nonzero commutative Noetherian ring. For every $H \leq G$ and $T \subseteq B$, put*

$$r_{H,T}(\Lambda) = \sum_{b \in T} |\Delta_{H,b}(\Lambda)|.$$

The section ring $M_H^T = N_H^T(\Lambda)$ is finite over a central Λ-algebra generated by $r_{H,T}(\Lambda)$ sections homogeneous in homological degree and the full formal twist multigrading. No Λ-subalgebra generated by fewer such homogeneous sections of nonzero twist can make M_H^T a finite module. This includes repeated and zero representations, with their distinct formal labels retained.

Proof. Theorem 7.8 gives the upper bound using the restricted dimension sums $\sigma_{b,d}$. We prove the matching lower bound, without assuming centrality of a proposed coefficient subalgebra.

Fix $b \in T$ and $d \in \Delta_{H,b}(\Lambda)$. If $d = \dim V_b$, choose a maximal ideal $\mathfrak{m}$ of Λ, put $k = \Lambda/\mathfrak{m}$, and take $P = 1$. Otherwise, choose a p-subgroup $P \leq H$ with $p \in \Pi_\Lambda$ and $\dim V_b^P = d$, and a maximal ideal of Λ containing p; its residue field k has characteristic p.

Projection to the face $\mathbb{N}e_b$ is a ring map: the sum of all other twist components is a two-sided ideal because twists are nonnegative. Follow this projection by coefficient base change to k, restriction to P, Brauer at P when $P \neq 1$, and forgetting the remaining group action. These operations induce a ring map

$$\phi_{b,d} : M_H^T \longrightarrow k[t], \qquad |t| = (d, e_b).$$

Indeed, the underlying Brauer sphere has one-dimensional homology in degree d, and compatible tensor powers of a fundamental class identify its one-label section algebra with $k[t]$. Multiplication here is concatenation; this assertion holds also when d is odd. For $P = 1$ use ordinary underlying homology. No isomorphism or surjectivity assertion for homology base change is required.

The restricted integral doubled orbit-sum forerunner at P maps to a nonzero scalar multiple of t^2. Consequently the image of $\phi_{b,d}$ is infinite-dimensional over k. A homogeneous section of degree (i, q), with $q \neq 0$, can have nonzero image only if $q = ne_b$ for some $n > 0$ and $i = nd$. If none of the generators of a proposed coefficient algebra has this degree type, that algebra has scalar image in $k[t]$. Module-finiteness would then make $\phi_{b,d}(M_H^T)$ finite-dimensional over k, a contradiction.

Thus a coefficient generator is required for every pair (b, d). One fully homogeneous section cannot serve two different pairs: its pure twist coordinate determines b, and its ratio i/n determines d. Hence at least $r_{H,T}(\Lambda)$ homogeneous generators of nonzero twist are necessary. The empty face has ring Λ and count zero, so is included. □

Remark 7.10. The exact count applies when the coefficient complex is Λ and the full formal twist multigrading is retained. It does not assert the same lower bound for an arbitrary coefficient complex D (which may be zero), for generators mixing homological degrees, or after collapsing formal twist coordinates. It is not the Krull dimension or a minimum number of algebra generators of the entire section ring.

For a single twist coordinate, the *homological slope* of a section of twist degree $q > 0$ and homological degree i is $i/q = -s/q$.

Proposition 7.11 (Optimal homogeneous coefficient count for the doubled regular representation)**.** *Let R_{reg} be the effective section ring for the single representation $2\mathbb{R}[G]$ over a nonzero commutative Noetherian ring Λ. It is finite over a central algebra with exactly*

$$r_\Lambda = 1 + \sum_{p \in \Pi_\Lambda} v_p(|G|)$$

displayed homogeneous generators, given by the geometric dimension sums of twist degree one. Any Λ-subalgebra generated by sections homogeneous for both the homological and twist gradings, and making R_{reg} a finite module, requires at least r_Λ homogeneous generators of twist degree $q > 0$. Over a field of characteristic p, this number is $v_p(|G|) + 1$.

Proof. For $V = \mathbb{R}[G]$, the fixed dimensions are $[G : P] = |G|/|P|$. A Sylow p-subgroup has subgroups of every order p^j, $0 \leq j \leq v_p(|G|)$, and distinct positive prime powers give distinct dimensions. The preceding theorem therefore gives exactly r_Λ dimension sums. They lie in the even-twist Veronese of the section ring for V. That Veronese is a submodule of a finite module over their Noetherian algebra, hence finite, and identifies with R_{reg} after rescaling the twist grading: choose the join model $C_{2\mathbb{R}[G]} = C_{\mathbb{R}[G]}^{\otimes 2}$. The doubled maps then have twist one and degree $2[G : P]$.

For the lower bound, fix $p \in \Pi_\Lambda$ and choose a maximal ideal $\mathfrak{m}$ containing p, so that $k = \Lambda/\mathfrak{m}$ is a residue field of characteristic p. For each p-subgroup P, coefficient base change, Brauer at P, and forgetting the Weyl action give a graded ring map

$$R_{\mathrm{reg}} \longrightarrow k[t], \qquad |t| = (2[G : P], 1).$$

Compatible top fundamental classes give the polynomial multiplication. The image of the twist-degree-one forerunner at P is a nonzero scalar times t, so the image is infinite-dimensional over k. No surjectivity of the entire homology base-change map is needed.

A homogeneous section of twist degree $q > 0$ and homological degree i can survive only when $i/q = 2[G : P]$. If no homogeneous generator of a proposed coefficient algebra has this slope, that algebra has scalar image in $k[t]$. Its infinite-dimensional quotient image cannot then be finite over it, contradicting module-finiteness. Thus every distinct fixed dimension requires a generator of a different slope. The trivial subgroup requires slope $2|G|$. If Π_Λ is empty, choose a maximal ideal $\mathfrak{m}$ and put $k = \Lambda/\mathfrak{m}$; ordinary underlying homology gives the same requirement. This proves the lower bound. □

Remark 7.12 (Deletion versus recombination). For the section ring of $2\mathbb{R}[G]$ over a field of characteristic p, the family consisting of one geometric forerunner per G-conjugacy class of p-subgroups is irredundant under deletion. Under the map to $k[t]$ obtained by Brauer at P, forgetting the Weyl action, and taking ordinary homology, the forerunner at Q vanishes unless Q is conjugate to P. Unequal orders give the wrong homological degree; equal orders and distinct conjugacy classes give distinct regular fixed subspaces and hence no fixed top simplex. Omitting the P-map therefore makes the remaining coefficient action scalar on an infinite image. In particular, the forerunners indexed by elementary abelian subgroups do not suffice when the p-group is not elementary abelian. The dimension sums improve this by recombining maps of the same degree. Optimality concerns the number of homogeneous coefficient generators needed for module-finiteness; it does not claim that the smaller algebra alone recovers the Balmer spectrum.

8 The geometric comparison

Until Subsection 8.3, let G be a finite p-group and let k be a field of characteristic p. We prove Corollary C for the canonical effective ring R_G. Miller introduced the comparison map and formulated its open-immersion and Dirac-scheme conclusions [20, Sections 5–6]. We first separate points by effective sections, using a fixed-sphere factor to isolate one term in normalizer transfer. We then use Noetherianity and central localization to identify the charts and their sheaves. This avoids the individual stable-class lifting assertion in [20, Proposition 6.1], which fails in the cited version by Example A.3.

8.1 Injectivity for the canonical family

For $H \leq G$, write $\Psi^H(C) = C(H)$ for termwise Brauer, with its residual $N_G(H)/H$-action, and write $\check{\Psi}^H$ for its composite with passage to $D^b(k[N_G(H)/H])$. For $W = N_G(H)/H$, the cohomological comparison identifies $\operatorname{Spc} D^b(kW)$ with $\operatorname{Spec}^h H^*(W;k)$ [3, Proposition 3.23 and its proof]. We use $\mathfrak{p}$ for a homogeneous cohomological prime and its corresponding prime tensor ideal; inverse images under $\check{\Psi}^H$ refer to the latter. We use the following established description of permutation spectra [3, Recollection 3.12, Proposition 3.23, Theorem 7.16, Corollaries 7.24 and 7.31]. Every point of X_G has the form

$$\mathscr{P}(H,\mathfrak{p}) = (\check{\Psi}^H)^{-1}(\mathfrak{p}), \qquad \mathfrak{p} \in \operatorname{Spec}^h H^*(N_G(H)/H;k),$$

uniquely up to conjugation. The stratum at 1 is the open spectrum of $D^b(kG)$, and the kernel of $\mathcal{K}(G)^\natural \to D^b(kG)$ is the ideal of acyclic complexes. Every point in the H-stratum specializes

to the closed point $\mathscr{M}(H)$ given by forgetting the residual group action after Brauer at H. We use also the classification of thick tensor ideals by support in a rigid category [1, Theorem 4.10]. Here $\operatorname{supp}(C) = \{\mathscr{P} : C \notin \mathscr{P}\}$, and the support of an ideal is the union of its objects' supports.

We construct Miller's comparison [20, Proposition 5.4 and Remark 5.12] directly. Put $d_{H,b} = \dim V_b^H$ and choose $\check{\Psi}^H(C_b) \simeq k[d_{H,b}]$, using its tensor powers for repeated factors. The orientation character is trivial over k for a p-group. There is a graded homomorphism

$$R_G \longrightarrow H^*(W;k)[x_b^{\pm 1} : b \in \mathcal{B}_G], \qquad f \longmapsto \check{\Psi}^H(f) \prod_b x_b^{q_b},$$

where $f : k \to C_{\mathcal{B}_G}(q)[s]$, $\deg(x_b) = (-d_{H,b}, e_b)$ and a class in $H^n(W;k)$ has degree $(n,0)$. The image $\check{\Psi}^H(f)$ has degree $s + \sum_b q_b d_{H,b}$. The multiplication signs agree: the $d_{H,b}$ are even at odd p, and signs vanish at $p = 2$. The extension of $\mathfrak{p}$ to this Laurent ring is fully homogeneous prime, since Laurent units reduce its homogeneous prime test to the one in $H^*(W;k)$. Its contraction defines

$$c_G(\mathscr{P}(H,\mathfrak{p})) = \langle f \text{ homogeneous} : \check{\Psi}^H(f) \in \mathfrak{p} \rangle. \tag{13}$$

Changing the chosen identifications multiplies homogeneous images by nonzero scalars, and conjugation transports the same prime; hence this definition is independent of those choices. The ordinary cohomological comparison gives

$$f \in c_G(\mathscr{P}) \quad \Longleftrightarrow \quad \operatorname{cone}(f) \notin \mathscr{P}.$$

Thus the inverse image of $D(f)$ is the open locus where f is invertible modulo the prime, proving continuity.

Lemma 8.1 (Regular-sphere support)**.** *Let $V = 2\mathbb{R}[G]$ and $U = V^H$. The setwise stabilizer of U is $N_G(H)$ and its pointwise stabilizer is H. The orbit-sum forerunner $\theta_{H,V}$ has zero Brauer map at K unless K is conjugate to a subgroup of H.*

Proof. In the left regular representation, the indicator of the subset $H \subseteq G$ is H-fixed, and its stabilizer is exactly H. Thus the pointwise stabilizer of the entire fixed subspace is H. Conjugating pointwise stabilizers now shows that the setwise stabilizer is $N_G(H)$. Doubling does not change either assertion.

Write $d = \dim U$. The cycle defining $\theta_{H,V}$ is supported on $(d-1)$-simplices in translates of $S(U)$, in chain degree d. If such a simplex is K-fixed, admissibility makes K fix it pointwise. The cone on the simplex has nonempty interior in the corresponding d-dimensional translate of U, so the simplex spans that translate. Thus K lies in its pointwise stabilizer, a conjugate of H. If no such containment holds, no basis simplex survives the Brauer projection. □

Lemma 8.2 (Effective numerators on a normal stratum)**.** *Let $H \trianglelefteq N$ be subgroups of a finite p-group. Distinct homogeneous primes of $H^*(N/H;k)$ are separated, under (13), by a section into an actual representation sphere for N.*

Proof. Put $W = N/H$, $V = 2\mathbb{R}[W]$, $D = \dim V$, and let $t : k[D] \to C_V$ be its top fundamental section. At the trivial subgroup it is a quasi-isomorphism. At each nontrivial $J \leq W$ its Brauer map is zero: $\dim V^J < D$, so the fixed-sphere complex has no term in degree D. At every point of a nontrivial J-stratum, the derived Brauer image of $\operatorname{cone}(t)$ thus contains the invertible summand $k[D+1]$; at the trivial stratum it vanishes. Its support is therefore exactly the support of the acyclic ideal, the complement of the cohomological stratum [3, Proposition 3.23 and

Corollary 7.24]. In a rigid category thick tensor ideals are radical, so support classification identifies the ideal generated by this cone with the acyclic ideal. Localization at t therefore gives $D^b(kW)$ by [3, Recollection 3.12].

Apply tensor localization with the invertible object $u = C_V[-D]$ and the map $t : k \to u$. Use $S = \{1, t^2, t^4, \ldots\}$ in the even part of the u-graded central ring. Inverting t^2 inverts t and has the same cone ideal. Choose a homogeneous class $\zeta : k \to k[n]$ in one of the two primes and outside the other. The fractions description [2, Theorem 3.6] says that every such class is represented by

$$f/t^m, \qquad f : k \longrightarrow C_V^{\otimes m}[n - mD], \qquad m \geq 0.$$

Here m may be chosen even, and the numerator is a morphism $k \to u^{\otimes m}[n]$ in the category *before* localization. The fully faithful embedding into its idempotent completion does not change maps between these original complexes. For $i = mD - n$,

$$\mathrm{Hom}_{\mathscr{K}(W)}(k[i], C_V^{\otimes m}) = H_i((C_V^{\otimes m})^W).$$

Thus f is represented by an actual W-invariant degree-i cycle. Since the separating class is nonzero and the chain complex is concentrated in degrees $0, \ldots, mD$, necessarily $0 \leq i \leq mD$. The sphere exponent is nonnegative. The denominator identifies $C_V^{\otimes m}$ with $k[mD]$, giving total shift n; being invertible at both points, it preserves the distinction. Inflation to N, followed by Brauer at the normal subgroup H, recovers the original complex and map: $\Psi^H \circ \mathrm{Inf}_W^N \cong \mathrm{Id}$ [3, Corollary 5.16]. The ordered join of the m sphere models is admissible, since the action preserves each factor, and its augmented chains are $C_V^{\otimes m}$ by Lemma 2.1. Thus the target is an actual sphere model for the inflated representation $V^{\oplus m}$. The inflated cycle distinguishes the two points on the H-stratum. □

Lemma 8.3 (Transfer with a fixed-sphere factor)**.** *Let $H \leq G$ and $N = N_G(H)$. A section into an actual N-sphere that separates two points on its H-stratum yields a section into an actual G-sphere separating the corresponding points of X_G.*

Proof. Represent the section by an N-invariant cycle f in an admissible model C_V. For the numerator from Lemma 8.2, use the ordered join model just described. Any subsequent change of admissible model is made by an actual equivariant chain homotopy equivalence, so the cycle and its separating property are preserved. The actual representation $Y = \mathrm{Ind}_N^G V$ satisfies

$$\mathrm{Res}_N^G Y = V \oplus V',$$

because the identity coset is N-stable. Multiply f by an H-forerunner for the actual N-representation V'. The admissible sphere-chain equivalence for a direct sum identifies this product with an N-invariant cycle f' in $\mathrm{Res}_N^G C_Y$. The equivalence is N-equivariant: the join of admissible triangulations, with its two factors preserved, is admissible, and common subdivision gives equivariant chain homotopy equivalences, as detailed in Lemma 2.1. Since the extra forerunner becomes an isomorphism after $\check{\Psi}^H$, this multiplication preserves separation.

Choose a G-admissible model $Q = C_{2\mathbb{R}[G]}$ compatible with all fixed subspaces. Put $U = (2\mathbb{R}[G])^H$ and $d = \dim U$. Choose a fundamental cycle $z_U \in Q_d$ of the unit sphere $S(U)$ in shifted degree d. It is N-invariant: the setwise stabilizer is N by Lemma 8.1, and its determinant character on U is the square of its determinant character on $\mathbb{R}[G]^H$, hence is trivial. The cycle $z = f' \otimes z_U$ is N-invariant in the restriction of the single global complex $C_Y \otimes Q$. Form the trace

$$F = \sum_{g \in G/N} gz.$$

It is a G-invariant cycle, with one common target and degree.

We compute its Brauer image at H on permutation bases. For an invariant cycle representing a section, Ψ^H denotes its fixed-basis image representing the Brauer map. If $g \notin N$, then $gU \neq U$ and their intersection has dimension less than d. A degree-d simplex in the support of gz_U cannot be H-fixed: it would lie in $S(gU \cap U)$, whose dimension is at most $d-2$. Thus the linear projection onto the H-fixed basis annihilates gz_U. Projection onto fixed bases commutes with tensor product, so every such trace term contributes zero. The identity coset gives

$$\Psi^H(F) = \Psi^H(f') \otimes z_U. \tag{14}$$

Individual nonidentity trace terms need not be H-invariant. We only use the basis projection to compute the Brauer image of the invariant sum; no chain-map assertion on arbitrary vectors is being made. The coefficient of the surviving term is one.

The last factor in (14) is a quasi-isomorphism onto the H-fixed sphere. Hence F has the desired separating property. Its target is the sphere of the actual representation $Y \oplus 2\mathbb{R}[G]$. Decomposing this representation into real irreducibles, and absorbing trivial summands into the shift, makes F a homogeneous element of the canonical effective ring. Representations with the same fixed-dimension function give isomorphic sphere objects by [19, Theorem 4.6]: their Brauer homology degrees agree, and both objects are invertible by Lemma 2.4. Choose an isomorphism for each irreducible summand and tensor these choices. The decomposition has only nonnegative multiplicities, so no negative twist is used. □

Proposition 8.4 (Injectivity of the comparison)**.** *The map $c_G : X_G \to \operatorname{Spec}^h(R_G)$ is injective.*

Proof. For nonconjugate H, K, at least one is not subconjugate to the other. Lemma 8.1 then gives a section whose derived Brauer image is an isomorphism at one subgroup and zero at the other. It separates every point of those two strata by (13). For two distinct points of the same H-stratum, apply Lemma 8.2 in $N_G(H)$, where H is normal, and then Lemma 8.3. This separates all possible pairs of distinct points. □

8.2 Charts and their sheaves

Lemma 8.5 (Comparison with finite endomorphism modules)**.** *Let $\mathcal{Q}$ be an essentially small rigid tensor triangulated category. Suppose that $A = \bigoplus_{s\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{Q}}(\mathbf{1}, \mathbf{1}[s])$ is Noetherian and that $\bigoplus_{s\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{Q}}(X, X[s])$ is finite over A for every $X \in \mathcal{Q}$. If its graded comparison map $\rho : \operatorname{Spc}(\mathcal{Q}) \to \operatorname{Spec}^h(A)$ is injective, it is a homeomorphism.*

Proof. This is the finite-endomorphism case of Lau's comparison argument [17, Proposition 2.7 and Corollary 2.8]; we give the proof. Put $E_X = \bigoplus_s \operatorname{Hom}_{\mathcal{Q}}(X, X[s])$. We claim that

$$\rho(\operatorname{supp}(X)) = \{\mathfrak{p} : (E_X)_{\mathfrak{p}} \neq 0\} = V(\operatorname{Ann}_A E_X).$$

The second equality follows from finite generation. For the first, central localization at the squares of the homogeneous elements outside $\mathfrak{p}$ gives a rigid category $\mathcal{Q}_{\mathfrak{p}}$ with localized morphism groups [2, Theorem 3.6]. The squares have even degree, and inverting them also inverts the original elements. The finite-endomorphism hypothesis survives this localization and idempotent completion, since localization preserves finite modules and morphism modules of retracts are direct summands. The image $X_{\mathfrak{p}}$ is zero exactly when $(E_X)_{\mathfrak{p}} = 0$, since this is equivalent to

the vanishing of its identity. If $\rho(\mathcal{P}) = \mathfrak{p}$, this localization maps to $\mathcal{Q}/\mathcal{P}$, so $X \notin \mathcal{P}$ implies $(E_X)_{\mathfrak{p}} \neq 0$.

Conversely, assume $X_{\mathfrak{p}} \neq 0$, and work in $\mathcal{Q}_{\mathfrak{p}}$. Its graded unit ring is the graded local ring $A_{\mathfrak{p}}$, with homogeneous maximal ideal $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}}$. For any nonzero object Y and homogeneous $f \in \mathfrak{m}$, the object $Y \otimes \operatorname{cone}(f)$ is nonzero. Indeed, otherwise $f : Y \to Y[|f|]$ would be an isomorphism, so $fE_Y = E_Y$. The nonzero finite graded $A_{\mathfrak{p}}$-module E_Y cannot satisfy this equality, by graded Nakayama. One can see this directly from a minimal homogeneous generating set: expressing one generator as f times a linear combination of the generators yields a coefficient $1 - fa$ of degree zero outside $\mathfrak{m}$, hence a unit, and eliminates that generator.

Rigidity implies $X_{\mathfrak{p}}^{\otimes n} \neq 0$ for every $n \geq 0$: a rigid tensor-nilpotent object is zero, by the duality retractions. Consequently the tensor-multiplicative family

$$\left\{ X_{\mathfrak{p}}^{\otimes n} \otimes \operatorname{cone}(f_1) \otimes \cdots \otimes \operatorname{cone}(f_r) : n, r \geq 0,\ f_i \in \mathfrak{m} \text{ homogeneous} \right\}$$

does not contain zero. The prime-existence lemma [1, Lemma 2.2] gives a prime disjoint from this family. It avoids $X_{\mathfrak{p}}$ and every $\operatorname{cone}(f)$ with $f \in \mathfrak{m}$; the cone criterion therefore makes its comparison prime equal to $\mathfrak{m}$. Its inverse image in $\mathcal{Q}$ avoids X and has comparison prime $\mathfrak{p}$. This proves the claim.

Taking $X = \mathbf{1}$ proves surjectivity. Finally, every closed subset of $\operatorname{Spc}(\mathcal{Q})$ is an intersection of supports of objects. Injectivity makes its image the corresponding intersection of the closed sets $V(\operatorname{Ann}_A E_X)$. Thus ρ is a continuous closed bijection, hence a homeomorphism. □

Sanders proves a more general criterion requiring only coherence of the graded unit ring in a rigidly compactly generated category [24, Theorem 10.8]. Here the bounded-coefficient theorem supplies the stronger finite-endomorphism hypothesis directly.

For the canonical family, fix the global forerunners $t_{H,b} = \theta_{H,b}$, and set

$$F_H = \prod_{b \in \mathcal{B}_G} t_{H,b}, \qquad L_H = R_G[t_{H,b}^{-1} : b \in \mathcal{B}_G], \qquad A_H = (L_H)_{\text{twist}=0}.$$

For the trivial group the product is one. All these denominators are central. Use suspension degree $s = -i$. Every twist component of L_H is a unit multiple of A_H, so multiplication induces an isomorphism, with the indicated full multigrading,

$$L_H \cong A_H[t_{H,b}^{\pm 1} : b \in \mathcal{B}_G]. \tag{15}$$

Injectivity follows because distinct Laurent exponents have distinct twists. In particular A_H is an ungraded algebra quotient of L_H by $t_{H,b} - 1$ and is Noetherian by Corollary B. Extension and contraction give inverse correspondences between homogeneous ideals of A_H and fully homogeneous ideals of L_H. Primality is preserved because multiplication by the units $t_{H,b}$ reduces the test to twist zero. On principal opens this yields a homeomorphism

$$\operatorname{Spec}^h_{\mathbb{Z}}(A_H) \cong D(F_H) \subseteq \operatorname{Spec}^h_{\mathbb{Z} \times \mathbb{Z}^{\mathcal{B}_G}}(R_G). \tag{16}$$

Let U_H be the common invertibility locus of the $t_{H,b}$ in X_G. It is the spectrum of the idempotent completion of the Verdier quotient by the thick tensor ideal generated by the cones of the $t_{H,b}$. Successively apply tensor localization to $u_b = C_b[-\dim V_b^H]$ and the maps $k \to u_b$ represented by $t_{H,b}$; one may invert their squares first. The fractions description [2, Theorem 3.6] identifies the suspension-graded endomorphism ring of the unit of this quotient with A_H: a

numerator into a sphere twist is divided by a product of the $t_{H,b}$ in the same twist. Thus its usual graded comparison is the restriction of c_G under (16). The opens U_H cover X_G. Indeed every $t_{H,b}$ is a quasi-isomorphism after Brauer at H, so $\mathscr{M}(H) \in U_H$; every point specializes to one of these closed points, and opens contain their generalizations.

Let $\mathcal{Q}_H$ be this idempotent-completed quotient. It is rigid: bounded permutation complexes have permutation duals, and duality passes to the tensor quotient and its idempotent completion. We verify the finite-endomorphism hypothesis of Lemma 8.5. For bounded permutation complexes X, Y, let $X^\vee = \mathrm{Hom}_k(X, k)$ be the chain dual, and put $D = X^\vee \otimes Y$ and

$$M_D = \bigoplus_{q \in \mathbb{N}^{\mathcal{B}_G},\ s \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{K}(G)}(k, C_{\mathcal{B}_G}(q) \otimes D[s]).$$

Theorem A makes M_D finite over R_G. The fractions description and duality give

$$\bigoplus_s \mathrm{Hom}_{\mathcal{Q}_H}(X, Y[s]) \cong \left(M_D[t_{H,b}^{-1} : b \in \mathcal{B}_G]\right)_{\mathrm{twist}=0}.$$

Choose finitely many homogeneous generators over L_H after localization. Multiplying each by a Laurent monomial in the $t_{H,b}$ moves its twist to zero. Those rescaled generators generate the displayed twist-zero part over A_H, by (15). Every object of $\mathcal{Q}_H$ is a retract of the image of a bounded permutation complex, and morphism modules between retracts are direct summands. They are therefore finite over A_H as well. The ring A_H is Noetherian, and Proposition 8.4 supplies injectivity. Lemma 8.5 now identifies $U_H = \mathrm{Spc}(\mathcal{Q}_H)$ with $\mathrm{Spec}^h(A_H)$. It follows that

$$c_G(X_G) = \bigcup_{H \le G} D(F_H)$$

and c_G is a homeomorphism onto this open subspace.

Define $\mathcal{O}_G^*$ by sheafifying, on the quasi-compact open basis,

$$U \longmapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{K}(G)|_U}(k, k[n]).$$

Here $\mathscr{K}(G)|_U$ is the idempotent completion of the Verdier quotient by the thick tensor ideal of objects supported in $X_G \setminus U$. Transitivity of Verdier localization identifies the restriction of this sheaf to U_H with the unit-endomorphism sheaf of the localized category. For a homogeneous $a \in A_H$, central localization identifies the presheaf on principal opens with the standard localization presheaf, whose value on $D(a)$ is $A_H[a^{-1}]$, with the usual restriction maps. One may use even a: $D(a) = D(a^2)$, while at odd characteristic odd elements have empty principal open. Sheafification therefore gives the standard affine graded sheaf [2, Lemma 6.9 and Proposition 6.11(b)]. By [11, Theorem 2.26 and Definition 2.29], these are affine Dirac schemes. They glue as restrictions of the single sheaf $\mathcal{O}_G^*$, proving the Dirac-scheme assertion in Corollary C. The ambient spectrum carries the full multigrading, whereas the Dirac charts carry the suspension grading.

We reuse this chart construction below for section rings in tensor-invertible sphere twists, whose invertibility follows from Lemma 7.1 after base change. Its other inputs are central homogeneous sections whose twists form the coordinate basis, the finite-module assertion of Theorem A for every bounded permutation coefficient complex, and an independently established injective comparison satisfying the cone criterion above. The finite-module assertion gives both the Noetherian section ring and, by the argument with $D = X^\vee \otimes Y$, finite graded morphism

modules over each twist-zero chart ring. The cone criterion identifies the localized comparison with the usual suspension-graded comparison. The Laurent description, tensor localization, Lemma 8.5, and the single sheaf $\mathcal{O}_G^*$ then give the affine Dirac charts. Coverage of the spectrum is checked separately in each application.

8.3 A Noetherian model with one twist

The argument admits a smaller choice of twists and a larger class of groups. From this subsection onward, let G be any finite group and let k be a field of characteristic $p > 0$, except where a statement imposes further hypotheses. Keep $\mathscr{K}(G) = K^b(\mathrm{perm}(kG))^\natural$ and $X_G = \mathrm{Spc}(\mathscr{K}(G))$. Put

$$U_G = 2\mathbb{R}[G], \qquad C = C_{U_G}, \qquad R_G^{\mathrm{reg}} = \bigoplus_{q\geq 0,\ s\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{K}(G)}(k, C^{\otimes q}[s]),$$

with full bidegree (s, q). For every p-subgroup $H \leq G$ choose the integral orbit-sum forerunner, reduced to k,

$$t_H : k[d_H] \longrightarrow C, \qquad d_H = 2[G : H].$$

Its bidegree is $(-d_H, 1)$. Doubling makes the orientation of every fixed space trivial under its full setwise stabilizer, so this map exists even when G is not a p-group.

Theorem 8.6 (A Noetherian model with one twist)**.** *The ring R_G^{reg} is Noetherian. Its comparison map*

$$c_G^{\mathrm{reg}} : X_G \longrightarrow \mathrm{Spec}^h_{\mathbb{Z}\times\mathbb{Z}}(R_G^{\mathrm{reg}})$$

is a homeomorphism onto $\bigcup_{H\leq G,\ H \text{ a } p\text{-group}} D(t_H)$. The ringed space $(X_G, \mathcal{O}_G^*)$ is a Dirac scheme. The corresponding affine chart rings are

$$A_H = (R_G^{\mathrm{reg}}[t_H^{-1}])_{\mathrm{twist}=0},$$

graded by suspension degree. The minimum number of elements homogeneous in both gradings whose generated k-subalgebra makes R_G^{reg} a finite module is $v_p(|G|) + 1$.

Together with Corollary 8.7, this proves Theorem D. Miller's research statement announces a comparison using reduced twisted cohomology and a Dirac conclusion for finite groups [23, Theorem 1.9]. Here Theorem A supplies an explicit Noetherian model with one effective twist.

Proof. We first define the comparison without using a canonical irreducible basis for G. By the general finite-group stratification [3, Theorem 7.16], every point is $\mathscr{P}(H, \mathfrak{p})$, where H is a p-subgroup and $\mathfrak{p} \in \mathrm{Spec}^h H^*(W; k)$ for $W = N_G(H)/H$, uniquely up to conjugation. The doubled fixed sphere has trivial orientation as a W-representation, so choose $\check{\Psi}^H(C) \simeq k[d_H]$ and use its tensor powers. Set

$$B_H = H^*(W; k)[x_H^{\pm 1}], \qquad \deg(\zeta) = (n, 0)\ (\zeta \in H^n(W; k)), \qquad \deg(x_H) = (-d_H, 1).$$

There is a bidegree-preserving homomorphism

$$\beta_H : R_G^{\mathrm{reg}} \longrightarrow B_H, \qquad f \longmapsto \check{\Psi}^H(f)x_H^q, \quad f : k \longrightarrow C^{\otimes q}[s],$$

since $\check{\Psi}^H(f)$ has cohomological degree $s + qd_H$. All d_H are even, so the multiplication signs agree. Define $c_G^{\mathrm{reg}}(\mathscr{P}(H, \mathfrak{p}))$ to be $\beta_H^{-1}(\mathfrak{p}B_H)$. The ideal $\mathfrak{p}B_H$ is fully homogeneous prime: multiplication

by Laurent units reduces the homogeneous prime test to $H^*(W;k)$. Rescaling an orientation or conjugating the pair does not change the contraction. Ordinary cohomological comparison gives

$$f \in c_G^{\mathrm{reg}}(\mathscr{P}) \quad \Longleftrightarrow \quad \mathrm{cone}(f) \notin \mathscr{P}.$$

Thus the map is continuous, with the usual invertibility loci as preimages of principal opens. The Laurent variable retains the full bidegree; no collapse of the two gradings is being used.

We prove injectivity. Lemma 8.1 applies to the present U_G: its fixed subspace at H has pointwise stabilizer H and setwise stabilizer $N_G(H)$. Its forerunner has zero K-Brauer map unless K is subconjugate to H, and is a quasi-isomorphism at H. Hence the t_H separate all points belonging to nonconjugate subgroup strata.

For two points on one H-stratum put $N = N_G(H)$ and $W = N/H$. The effective-numerator argument of Lemma 8.2 works for this arbitrary finite W, with doubled target, as follows. Take $V = 2\mathbb{R}[W]$, $D = \dim V$, and its top section $t : k[D] \to C_V$. Its ordinary image is a quasi-isomorphism, and its Brauer image at every nontrivial p-subgroup is zero. Its cone has precisely the support of the acyclic ideal, by [3, Proposition 3.23 and Corollary 7.24]. Rigidity and support classification identify the generated ideal with that acyclic ideal; its Verdier quotient is $D^b(kW)$ by [3, Recollection 3.12]. Tensor localization [2, Theorem 3.6], applied to $C_V[-D]$ and t (or t^2), represents a homogeneous cohomology class separating the two primes by

$$f/t^m, \qquad f : k \longrightarrow C_V^{\otimes m}[n - mD], \qquad m \geq 0.$$

Inflation to N, followed by Brauer at H, recovers this map [3, Corollary 5.16]. Thus we have a separating section into an actual doubled N-representation $2V_0$.

The transfer proof also extends, with doubling retained. Put $Y = \mathrm{Ind}_N^G(2V_0)$; its restriction decomposes as $2V_0 \oplus 2V_0'$. Multiply the section by the integral H-forerunner for $2V_0'$, and use the N-equivariant join/subdivision equivalence to obtain an N-invariant cycle f' in $\mathrm{Res}_N^G C_Y$. This multiplication preserves separation because its H-Brauer image is invertible. For $E = U_G^H$, let z_E be a top fundamental cycle of $S(E)$ in the shifted augmented chain complex C. It is N-invariant integrally, since E is doubled. The trace

$$F = \sum_{g \in G/N} g(f' \otimes z_E)$$

is a global cycle in $C_Y \otimes C$. If $g \notin N$, no top simplex of $gS(E)$ is H-fixed, since $gE \cap E$ has smaller dimension. Projection onto fixed permutation bases therefore gives

$$\Psi^H(F) = \Psi^H(f') \otimes z_E.$$

As in Lemma 8.3, the projection computes Brauer of the invariant sum even though individual summands need not be invariant. Exactly one term survives. In particular this argument does not divide by $[G : N]$. The section F separates the two original points and has an actual doubled G-target $2T$.

Every real irreducible occurs in $\mathbb{R}[G]$. Choose an actual representation Z and an integer $r \geq 1$ with $T \oplus Z \cong r\mathbb{R}[G]$. Multiplication by the global integral H-forerunner for $2Z$ preserves separation at both points and places the target in

$$C_{2T \oplus 2Z} \simeq C^{\otimes r}.$$

The actual doubled complement places the separator in a positive twist of R_G^{reg} and preserves the required orientation invariance. It follows that c_G^{reg} is injective.

Theorem A applied to U_G makes R_G^{reg} Noetherian and supplies the bounded-coefficient finiteness needed for the chart construction. Proposition 7.11 gives the asserted optimal homogeneous coefficient count. The coefficient generators of twist degree two supplied by Theorem A need not be squares of the chosen t_H, which have twist degree one. The t_H are central, and multiplication gives

$$R_G^{\mathrm{reg}}[t_H^{-1}] \cong A_H[t_H^{\pm 1}], \qquad \mathrm{Spec}_{\mathbb{Z}}^h(A_H) \cong D(t_H).$$

Indeed, every twist component is a Laurent-unit multiple of twist zero. Evaluation at $t_H = 1$ makes A_H an ungraded algebra quotient, so it is Noetherian. Extension and contraction preserve homogeneous primality by translating to twist zero.

Let $\mathscr{K}_H$ be the idempotent-completed Verdier quotient by the thick tensor ideal generated by $\mathrm{cone}(t_H)$. Its spectrum is the invertibility locus U_H of t_H. The chart construction above applies to the Laurent description just proved and the independently established injectivity of c_G^{reg}. It identifies the suspension-graded unit ring of $\mathscr{K}_H$ with A_H and gives an affine Dirac chart $U_H \cong \mathrm{Spec}^h(A_H) \cong D(t_H)$, whose sheaf is the restriction of $\mathcal{O}_G^*$. Every point on the H-stratum specializes to $\mathscr{M}(H)$ [3, Corollary 7.31], which belongs to U_H; hence these opens cover X_G. This proves the stated open image and homeomorphism. The full bidegree describes the topological immersion, while suspension degree describes the Dirac charts.

Explicitly, on an overlap put $a_{HK} = t_K/t_H \in A_H$, of suspension degree $d_H - d_K$. Then $U_H \cap U_K$ corresponds to $D(a_{HK})$ in the H-chart, and

$$A_H[a_{HK}^{-1}] \cong \left(R_G^{\mathrm{reg}}[t_H^{-1}, t_K^{-1}]\right)_{\mathrm{twist}=0} \cong A_K[(t_H/t_K)^{-1}].$$

These identifications are induced by localization. On triple overlaps they factor through the same localization at all three forerunners, so the transition maps satisfy the cocycle condition. □

Write $\mathrm{Sub}_p(G)$ for the set of all p-subgroups of G, including 1, with G acting by conjugation.

Corollary 8.7 (One extra point and an optimal principal cover)**.** *Let $a = v_p(|G|)$ and $R = R_G^{\mathrm{reg}}$. For $0 \le j \le a$, put*

$$d_j = 2|G|/p^j, \qquad \sigma_j = \sum_{\substack{[H] \in G\backslash \mathrm{Sub}_p(G) \\ |H| = p^j}} t_H, \qquad R_+ = \bigoplus_{q>0,\ s\in\mathbb{Z}} R_{s,q}.$$

Here the forerunners are chosen with the compatible integral orientations used for the dimension sums. Then

$$c_G^{\mathrm{reg}}(X_G) = \mathrm{Spec}_{\mathbb{Z}\times\mathbb{Z}}^h(R) \setminus \{R_+\} = \bigcup_{j=0}^{a} D(\sigma_j).$$

Thus the full homogeneous spectrum has exactly one point outside X_G, and this point lies in the closure of every point of X_G. The $a+1$ opens $D(\sigma_j)$ have affine Dirac chart rings

$$A_j = \left(R[\sigma_j^{-1}]\right)_{\mathrm{twist}=0},$$

graded by suspension degree. Moreover, $a+1$ is the minimum number of fully homogeneous principal opens of the ambient spectrum, each contained in $c_G^{\mathrm{reg}}(X_G)$, that cover this image.

Proof. For the regular representation, equal fixed-space dimensions are equivalent to equal subgroup orders, and the fixed subspace determines its pointwise stabilizer. Consequently the σ_j are precisely the dimension sums of twist degree one in Proposition 7.11. In particular, R is finite over $k[\sigma_0, \dots, \sigma_a]$. It follows that $R/(\sigma_0, \dots, \sigma_a)$ is finite-dimensional over k. Its twist-zero part is k, because $\mathrm{Hom}_{\mathscr{K}(G)}(k, k[s])$ is k for $s = 0$ and zero otherwise. All elements of nonzero twist in this quotient are nilpotent: its nonnegative twist grading has finite support. Hence its unique fully homogeneous prime is the ideal generated by its nonzero twist components, and

$$\sqrt{(\sigma_0, \dots, \sigma_a)} = R_+.$$

Also $(\sigma_0, \dots, \sigma_a) \subseteq (t_H : H \in \mathrm{Sub}_p(G)) \subseteq R_+$, so the latter ideal has the same radical. Theorem 8.6 gives the two asserted descriptions of the image. Since every fully homogeneous proper ideal of R has zero twist-zero part, every homogeneous prime is contained in R_+. This proves the specialization assertion.

Each σ_j has twist one and even suspension degree. The chart argument in Theorem 8.6 applies verbatim: $R[\sigma_j^{-1}] = A_j[\sigma_j^{\pm 1}]$, A_j is Noetherian, and tensor localization identifies it with the suspension-graded endomorphisms of the unit in the idempotent-completed Verdier quotient by the thick tensor ideal generated by $\mathrm{cone}(\sigma_j)$. Theorem A with bounded permutation coefficients gives finite graded morphism modules over A_j by the preceding Laurent-unit argument. The restricted comparison is injective, so Lemma 8.5 gives the affine chart, and central localization identifies its Dirac sheaf.

Finally, a nonempty fully homogeneous principal open $D(f)$ contained in the image must have f of twist degree $q > 0$, since a nonzero twist-zero element is a scalar and its principal open contains R_+. Write f as a section $k[i] \to C^{\otimes q}$. At $\mathscr{M}(H)$, forgetting the residual action after Brauer identifies the target homology with $k[qd_j]$, where $|H| = p^j$. Thus f can be invertible there only if $i/q = d_j$. A single fully homogeneous principal open can therefore contain closed points from at most one subgroup order. A Sylow p-subgroup has subgroups of all orders p^j, $0 \le j \le a$, giving the lower bound $a + 1$. The displayed cover attains it. □

8.4 Coefficient bounds and contraction

Corollary 8.7 concerns principal opens in the spectrum of the full section ring. We now compare module-finiteness over a coefficient subalgebra with injectivity of the comparison after contraction to that subalgebra. We first give obstructions to separating the closed subgroup points. For cyclic p-groups, we then construct a coefficient subring with the optimal homogeneous generator count whose comparison is a homeomorphism onto the complement of one closed point.

Lemma 8.8 (Contraction to optimal coefficient algebras)**.** *Put $a = v_p(|G|)$. Suppose $S = k[f_0, \dots, f_a] \subseteq R = R_G^{\mathrm{reg}}$, where the f_j are homogeneous in both gradings, have nonzero twist degree, and R is finite over S. Then any two closed subgroup points $\mathscr{M}(H)$ of the same subgroup order have the same contracted prime on S.*

Proof. The slope lower bound in Proposition 7.11 forces, after reindexing, exactly one generator f_j of slope $d_j = 2|G|/p^j$. Write its twist as $q_j > 0$.

At each closed point $\mathscr{M}(H)$ with $|H| = p^j$, Brauer followed by ordinary homology gives $R \to k[t]$, $|t| = (-d_j, 1)$. All f_ℓ with $\ell \neq j$ vanish by degree. The image of R is infinite-dimensional because its H-forerunner survives. Module-finiteness therefore forces f_j to map to $c_H t^{q_j}$ with $c_H \neq 0$. All the f_j have even suspension degree, so the presentation map $k[Y_0, \dots, Y_a] \to S$ is an

ordinary polynomial algebra map. Its composite with $R \to k[t]$ has kernel exactly $(Y_\ell : \ell \neq j)$: the remaining map $k[Y_j] \to k[t]$, $Y_j \mapsto c_H t^{q_j}$, is injective. Hence the contracted prime on S is independent of H within that subgroup order. □

For $G = C_p \times C_p$, the $p+1$ distinct order-p subgroups therefore give distinct closed points with identical contractions to every optimal coefficient algebra. The dimension sums still give the optimal principal cover inside the spectrum of the full ring R; this cover is different from recovering X_G by contraction to the coefficient algebra itself.

Proposition 8.9 (Bounds under module-finiteness and closed-point separation)**.** *Let G be a finite group, let $k = \mathbb{F}_q$ have characteristic p, and put $a = v_p(|G|)$. For $0 \le j \le a$, let c_j be the number of G-conjugacy classes of subgroups of order p^j and set*

$$\mu_q(c) = \min\left\{ m \ge 1 : c \le \frac{q^m - 1}{q - 1} \right\}.$$

Suppose a subalgebra $S \subseteq R_G^{\mathrm{reg}}$ is generated by sections homogeneous in both gradings and of nonzero twist degree, R_G^{reg} is finite over S, and contraction of the comparison map to $\mathrm{Spec}^h(S)$ separates the closed points $\mathscr{M}(H)$. Then every such generating set has at least

$$\sum_{j=0}^{a} \mu_q(c_j)$$

elements. The generators may have different nonzero twist degrees.

Proof. Choose an ordinary homology generator for the Brauer image of $C_{2\mathbb{R}[G]}$ at each p-subgroup H. Its tensor powers give a graded algebra map

$$\epsilon_H : R_G^{\mathrm{reg}} \longrightarrow k[t], \qquad |t| = (-2[G:H], 1),$$

whose kernel is the comparison prime of $\mathscr{M}(H)$. The regular-sphere forerunner at H has nonzero image, so this image is infinite-dimensional over k. Module-finiteness therefore forces at least one coefficient generator to have nonzero image. Such a generator of twist $w > 0$ must have suspension degree $-2[G:H]w$. Thus different subgroup orders require disjoint blocks of generators.

Fix j, and suppose its slope block contains m generators of twist degrees $w_1, \dots, w_m > 0$. At a subgroup H of order p^j their images have the form $b_{H,i} t^{w_i}$ with $b_{H,i} \in k$, and the vector $(b_{H,1}, \dots, b_{H,m})$ is nonzero. Define

$$\rho_H : k[Y_1, \dots, Y_m] \longrightarrow k[t], \qquad Y_i \longmapsto b_{H,i} t^{w_i}, \qquad I_H = \{ i : b_{H,i} \neq 0 \}.$$

The set I_H is the support of the coefficient vector. The generators in this block have even suspension degree, so they commute. The contraction to S is determined by $\ker(\rho_H)$ together with the generators outside this slope block, all of which vanish.

Fix a nonempty coefficient support $I \subseteq \{1, \dots, m\}$. Put $d_I = \gcd(w_i : i \in I)$ and $v_i = w_i / d_I$. Replacing t^{w_i} by u^{v_i} does not change the polynomial kernel: the substitution $u \mapsto t^{d_I}$ is injective on $k[u]$, since distinct monomials have distinct exponents, even when $p \mid d_I$. For $\lambda \in k^\times$, replacing b_i by $\lambda^{v_i} b_i$ also preserves this kernel, by the automorphism $u \mapsto \lambda u$. This action on $(k^\times)^I$ is free: if $\lambda^{v_i} = 1$ for every i, Bézout's identity gives $\lambda = 1$. Each orbit is contained in one fiber of the kernel assignment, so there are at most $(q-1)^{|I|-1}$ kernels arising from coefficient

vectors with support I. No converse classification of kernels is needed. Summing over the nonempty supports gives at most

$$\sum_{r=1}^{m} \binom{m}{r} (q-1)^{r-1} = \frac{q^m - 1}{q-1}$$

possible contractions at this slope. This count takes place over k itself; no freeness is asserted for $(b_i) \mapsto (\lambda^{w_i} b_i)$ before dividing the weights by their gcd. Changing the chosen homology generator only rescales the evaluation map and does not affect its kernel.

The c_j conjugacy classes give distinct closed points $\mathscr{M}(H)$ of X_G. Injectivity of contraction therefore requires $c_j \leq (q^m - 1)/(q-1)$ for every j. Summing the lower bounds for the disjoint slope blocks proves the result. □

Remark 8.10. For $G = C_p^n$, the subgroup count is the Gaussian coefficient

$$c_j = \begin{bmatrix} n \\ j \end{bmatrix}_p = \prod_{r=0}^{j-1} \frac{p^n - p^r}{p^j - p^r}.$$

The bound is four in rank two over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$, eight in rank three over either field, and respectively sixteen and twelve in rank four. For example, in rank four the line and hyperplane counts are $p^3 + p^2 + p + 1$, and the plane count is $p^4 + p^3 + 2p^2 + p + 1$; comparison with $(q^m - 1)/(q-1)$ gives the stated bounds. These are necessary counts for simultaneous module-finiteness and separation of the closed subgroup points; module-finiteness alone requires only $n+1$ coefficients. The explicit constructions and their sharpness questions belong to the companion; no such construction is needed for the obstruction above.

Remark 8.11 (Proper subrings preserving all charts). Put $R = R_G^{\mathrm{reg}}$. For every $m \geq 2$, the twist Veronese

$$R^{(m)} = \bigoplus_{q \geq 0,\ s \in \mathbb{Z}} R_{s,mq}$$

is a proper subring of R that preserves the comparison and its Dirac charts, after rescaling the twist degree. With this rescaling, the sphere tensor equivalence identifies it with the section ring for mU_G, so Theorem A gives Noetherianity and bounded-coefficient finiteness. Indeed,

$$\left(R^{(m)}[(t_H^m)^{-1}]\right)_{\mathrm{twist}=0} = \left(R[t_H^{-1}]\right)_{\mathrm{twist}=0} = A_H.$$

To see the equality, multiply the numerator and denominator of f/t_H^q by t_H^b, where $b \geq 0$ makes $q + b$ divisible by m. The same localization and sheaf arguments therefore give the same open model of X_G from $R^{(m)}$. Properness follows because R has nonzero twist-one sections. Thus proper inclusion alone does not specify a useful minimality criterion.

8.5 An explicit subring for cyclic groups

For $G = C_{p^n}$ with $n \geq 1$, R_G^{reg} contains a homogeneous subring with the following presentation. Its spectrum of primes homogeneous in both gradings is the cyclic permutation spectrum computed by Balmer–Gallauer [3, Proposition 8.3], together with one additional closed point. The displayed quotient is the Stanley–Reisner ring of the path with vertices $0, \ldots, n$ and edges $\{t-1, t\}$ for $1 \leq t \leq n$, with the bidegrees specified below. The identity below is an arbitrary-prime chain-level analogue of the gold relation of Hill–Hopkins–Ravenel [12, Lemma 3.6(7)];

compare the Green-functor formulation in the treatment of cyclic 2-groups in [25, Proposition 3.16]. Basu and Dey compute constant-coefficient Euler–orientation relations for arbitrary cyclic groups [4, Section 1.7 and equations (7.1), (7.8)]. For odd-order cyclic groups, Dugger and Hazel give an algebraic treatment over the constant integral Mackey ring [6, Proposition 1.5 and Section 4.17]. We give the required constant-coefficient chain boundary explicitly.

Proposition 8.12 (A Stanley–Reisner subring for cyclic groups)**.** *Let $G = C_{p^n}$ with $n \geq 1$, and let k have characteristic p. The section ring R_G^{reg} contains a homogeneous subring*

$$S_G \cong k[X_0, \ldots, X_n]/(X_i X_j : 0 \leq i < j \leq n,\ j - i \geq 2), \qquad \deg(X_t) = (-2p^t, 1),$$

over which R_G^{reg} is a finite module. Contraction of the comparison map induces a homeomorphism

$$X_G \xrightarrow{\sim} \mathrm{Spec}^h_{\mathbb{Z}\times\mathbb{Z}}(S_G) \setminus \{(X_0, \ldots, X_n)\}.$$

The $n+1$ displayed generators attain the optimal homogeneous coefficient count of Proposition 7.11.

Proof. Sphere models. Fix a generator $G = \langle g \rangle$ and put $H_t = \langle g^{p^t} \rangle$ for $0 \leq t \leq n$, so $H_0 = G$ and $H_n = 1$. For $1 \leq i \leq n$, write $A_i = k[G/H_i] = k[C_{p^i}]$. Use g also for its image in every quotient, take the bases g^r in both copies of A_i, and let ϵ be the augmentation. Let E_i be the inflated rotation-sphere complex

$$E_i = (A_i \xrightarrow{g-1} A_i \xrightarrow{\epsilon} k)$$

in homological degrees $2, 1, 0$. This is a permutation homotopy model for the shifted augmented chains of the unit sphere of the two-dimensional rotation representation, including $p = 2, i = 1$, as follows. Subdivide the cellular edge $e_r : v_r \to v_{r+1}$ at a vertex w_r into $a_r : v_r \to w_r$ and $b_r : w_r \to v_{r+1}$, with indices modulo p^i. The resulting $2p^i$-gon is an admissible simplicial circle. In its augmented chains, put $c_r = a_r + b_r$ and $u_r = w_r - v_r$. Then

$$\partial c_r = v_{r+1} - v_r, \qquad \partial a_r = u_r, \qquad \epsilon(u_r) = 0.$$

These integral basis changes commute with cyclic translation and inflation. They split off contractible pairs $a_r \mapsto u_r$, leaving E_i. The top norm cycle is carried to $\sum_r c_r$. At $p = 2, i = 1$, the circle is the sphere of twice the sign representation. Let $a_i : k \to E_i$ be the bottom identity, and let $b_i : k[2] \to E_i$ be the top norm cycle $N_i = \sum_{r=0}^{p^i-1} g^r$.

Set $m_i = \varphi(p^i) = p^{i-1}(p-1)$. In the permutation homotopy category there is an isomorphism

$$C_{2\mathbb{R}[G]} \simeq k[2] \otimes \bigotimes_{i=1}^{n} E_i^{\otimes m_i}. \tag{17}$$

The decomposition of the real regular representation, followed by doubling, gives m_i rotations with quotient order p^i, and two trivial dimensions. Although faithful rotations need not be isomorphic as real representations, their complexes over k are identified by the following change of basis. A rotation has cellular differential $g^r - 1 = (g-1)h_r$ for an attachment exponent r prime to p. Since $\epsilon(h_r) = r \neq 0$ in k, the element h_r is a unit in $k[C_{p^i}]$. Changing the top basis by this unit identifies its complex with E_i. Moreover, $h_r N_i = r N_i$, so the top norm cycle is rescaled by a nonzero scalar. For $p = 2$, the doubled sign is one copy of E_1, and $m_1 = 1$ as

required. These chain identifications and the sphere tensor equivalence prove (17). Choose a chain homotopy inverse

$$\alpha : k[2] \otimes \bigotimes_{i=1}^{n} E_i^{\otimes m_i} \longrightarrow C_{2\mathbb{R}[G]},$$

and use its tensor powers for the ring construction.

Coefficient generators and module-finiteness. Define

$$x_t = \alpha \circ \left(\mathrm{id}_{k[2]} \otimes \bigotimes_{i \le t} b_i^{\otimes m_i} \otimes \bigotimes_{i > t} a_i^{\otimes m_i} \right), \qquad 0 \le t \le n. \tag{18}$$

Its homological degree is $2 + 2\sum_{i \le t} \varphi(p^i) = 2p^t$, so its full bidegree is $(-2p^t, 1)$. Under Brauer at H_t, the factors b_i for $i \le t$ are quasi-isomorphisms, whereas for $i > t$ both positive-degree terms of E_i disappear and a_i becomes the identity of k. Thus x_t is a forerunner at H_t. These are all the subgroups of G. Proposition 6.2, applied to $C_{2\mathbb{R}[G]}$ and these supplied maps, proves that R_G^{reg} is finite over $k[x_0, \dots, x_n]$.

Nonadjacent products. We exhibit $a_i b_j$ as a boundary when $i < j$. Put $m = p^i$, $M = p^j$, and use the compatible bases $e_r = g^r$ of A_i and $f_s = g^s$ of A_j. In each pair of positive degrees consider the invariant orbit sums

$$O_r = \sum_{t=0}^{M-1} e_{t \bmod m} \otimes f_{t+r \bmod M}, \qquad r \in \mathbb{Z}/m.$$

Let U_r, W_r, V_r denote this sum in tensor-factor degrees $(1,2), (2,1)$ and $(1,1)$ of $E_i \otimes E_j$, respectively. The tensor differential gives

$$\partial U_r = a_i b_j + V_r - V_{r+1}, \qquad \partial W_{r+1} = V_r - V_{r+1} + (M/m) b_i a_j.$$

Consequently

$$\partial(U_r - W_{r+1}) = a_i b_j - (M/m) b_i a_j = a_i b_j \tag{19}$$

over k. The displayed formulas also give the integral gold relation before reduction. If $u \ge t+2$, the product $x_t x_u$ has the factor $a_{t+1} b_{t+2}$, so it is zero.

No additional relations. For $0 \le t < n$, apply Brauer at H_{t+1} followed by passage to the derived category of $Q = G/H_{t+1} = C_{p^{t+1}}$. The image of x_{t+1} is an isomorphism. Relative to this isomorphism, the image of x_t is a nonzero scalar times $v^{m_{t+1}}$, where $v \in H^2(Q; k)$ is the class of the extension below, and $N_Q : k \to kQ$ is defined by $N_Q(1) = \sum_{h \in Q} h$:

$$0 \longrightarrow k \xrightarrow{N_Q} kQ \xrightarrow{g-1} kQ \xrightarrow{\epsilon} k \longrightarrow 0.$$

Indeed, exactly the E_{t+1} factors change from their top sections to their bottom sections; all other factors either give the same isomorphism or become k. Repeating multiplication by $g-1$ and by $\sum_{h \in Q} h$ gives a two-periodic free resolution of k. Applying $\mathrm{Hom}_{kQ}(-, k)$ gives zero differentials; the degree-two shift map represents v, and its iterates represent the nonzero classes v^r for every $r \ge 1$. For $Q = C_2$ in characteristic two, v is the square of the degree-one polynomial class, and the conclusion is unchanged. Every monomial in x_t, x_{t+1} is therefore nonzero.

The standard monomials of the claimed quotient have support in one vertex or one adjacent pair. On an adjacent pair the degree vectors $(-2p^t, 1)$ and $(-2p^{t+1}, 1)$ are linearly independent.

Monomials supported on different adjacent pairs have slopes in disjoint intervals $(2p^t, 2p^{t+1})$, apart from the common pure endpoint. Hence distinct standard monomials have distinct full bidegrees. Since every such monomial is nonzero, their images are linearly independent. This proves the presentation of S_G.

The induced map on spectra. Each nonzero homogeneous component of S_G is one-dimensional, so every homogeneous prime is monomial. The variables outside a prime must form a face of the path. Its homogeneous primes are exactly

$$\begin{aligned}
\mathfrak{p}_t &= (X_j : j \neq t) \quad (0 \leq t \leq n),\\
\mathfrak{q}_t &= (X_j : j \notin \{t-1, t\}) \quad (1 \leq t \leq n),\\
\mathfrak{m} &= (X_0, \ldots, X_n).
\end{aligned}$$

The nontrivial inclusions away from $\mathfrak{m}$ are $\mathfrak{q}_t \subset \mathfrak{p}_{t-1}, \mathfrak{p}_t$.

The H_t-stratum of X_G has its closed point $\mathscr{M}(H_t)$, and for $t \geq 1$ one additional point $\eta(H_t)$, because the cohomology of a nontrivial cyclic p-group has exactly two homogeneous primes. At $\mathscr{M}(H_t)$ only x_t survives, by its degree and forerunner property, giving $\mathfrak{p}_t$. At $\eta(H_t)$, the periodicity calculation shows that both x_t and x_{t-1} survive. Nonadjacent variables vanish by their product relations with x_t; the remaining possible variable x_{t+1}, when present, maps to negative cohomological degree $2p^t - 2p^{t+1}$ and is zero. Thus $\eta(H_t)$ gives $\mathfrak{q}_t$. These are precisely the points and specialization relations of the cyclic permutation spectrum [3, Proposition 8.3]. The continuous bijection just computed is therefore a homeomorphism of these finite spaces. □

Remark 8.13. The proposition gives generators and relations for a subring recovering the spectrum in the sense of [23, Question 1.11], for every cyclic p-group. To regard it as a subring of the canonical remixed ring, absorb the trivial $k[2]$ factor in (17) into the suspension degree: (s, q) becomes $(s + 2q, q)$. The remaining targets lie on the effective diagonal with exponents qm_i; at $p = 2$ the canonical sign coordinate instead has exponent $2q$. This diagonal has a nonzero exponent vector, so distinct twist degrees in R_G^{reg} remain distinct canonical multidegrees. Using tensor powers of the fixed sphere equivalence gives an injective ring map with this regrading. It does not present the full ring R_G^{reg}. The full bidegree is essential. We do not identify the sheaf obtained from S_G, which is concentrated in even suspension degrees, with the suspension-graded Dirac sheaf of Theorem D: at odd characteristic the latter retains odd cohomology classes omitted by S_G.

Corollary 8.14 (Optimal coefficient count and comparison injectivity)**.** *Let G be a nontrivial finite p-group of order p^n, and let k have characteristic p. There exists a subalgebra $S \subseteq R_G^{\mathrm{reg}}$ generated by $n+1$ fully homogeneous sections of nonzero twist, over which R_G^{reg} is finite, such that*

$$X_G \longrightarrow \mathrm{Spec}^h(S), \qquad \mathscr{P} \longmapsto c_G^{\mathrm{reg}}(\mathscr{P}) \cap S$$

is injective, if and only if G is cyclic. In the cyclic case, S can be chosen so that this map is a homeomorphism onto the complement of the ideal generated by its nonzero twist components.

Proof. Proposition 8.12 proves the cyclic case. If G is not cyclic, it has at least two distinct maximal subgroups: otherwise an element outside its unique maximal subgroup would generate G, since every proper subgroup is contained in a maximal one. Maximal subgroups of a finite p-group are normal of index p. Thus two such subgroups are nonconjugate and have the same order. Their distinct closed points have identical contractions to every optimal coefficient algebra by Lemma 8.8. This contradicts injectivity. □

A Restriction and multiplicative lifting

Let G be a finite p-group and let k be a field of characteristic p. The classes of the sphere objects C_b, $b \in \mathcal{B}_G$, together with the class of $k[1]$, form the canonical basis in [19, Theorem 4.12]. These sphere objects are called irreducible endotrivials in [20, Definition 2.13 and Notation 2.15]. Following [20, Definition 7.1], a bounded permutation kG-complex C is *res-onto* if, for every $H \leq G$ and every integer i, restriction induces a surjection

$$H_i(C^G) \longrightarrow H_i(C^H).$$

Here C^H means the termwise invariant complex; equivalently these are the restriction maps on morphisms from shifts of the unit in the permutation homotopy categories. The group G is res-onto if every C_b, $b \in \mathcal{B}_G$, is res-onto. This property depends only on the homotopy object, and hence not on the chosen sphere triangulation or representative of a repeated dimension function.

If $H \trianglelefteq G$ and C is a bounded permutation kG-complex, the action of G on C^H induces a G/H-action on $H_i(C^H)$, equivalently on $\mathrm{Hom}_{\mathcal{K}(H)}(k[i], \mathrm{Res}^G_H C)$. We call a section class *G-stable* if it is fixed by this action. Fixedness of the class need not give a G-invariant cycle representative. We first classify res-onto irreducible spheres and then give a stable class that does not lift. For a finite p-group P, let $\Phi(P) = P^p[P,P]$ be its Frattini subgroup, where P^p is generated by the pth powers; in particular $\Phi(1) = 1$.

Proposition A.1 (The next-to-top restriction obstruction)**.** *Let Q be a finite p-group, let k have characteristic p, and let V be a faithful real Q-representation of dimension $d \geq 2$. After choosing a nonzero top fundamental class of C_V, there are identifications, natural under subgroup restriction,*

$$H_{d-1}(C_V^L) \cong H^1(L;k) \qquad (L \leq Q).$$

Under these identifications the restriction map is ordinary cohomological restriction. In particular,

$$\dim_k \mathrm{coker}\big(H_{d-1}(C_V^Q) \longrightarrow H_{d-1}(C_V^L)\big) = \dim_{\mathbb{F}_p} \frac{L \cap \Phi(Q)}{\Phi(L)}.$$

Proof. Choose an admissible sphere triangulation and write $X = C_V$ for its shifted augmented chain complex. Set $Z_j(X) = \ker(\partial : X_j \to X_{j-1})$. By admissibility, a top-simplex stabilizer fixes an open subset of the sphere and hence the entire representation. Faithfulness makes this stabilizer trivial, so the top chain term X_d is free over kQ. The orientation line of a real representation of a p-group is trivial over k. Identify it with k using the chosen fundamental class. Ordinary sphere homology then gives

$$0 \longrightarrow k \longrightarrow X_d \longrightarrow Z_{d-1}(X) \longrightarrow 0.$$

The module X_d restricts to a free, hence injective, kL-module for every L. The connecting homomorphism therefore gives

$$H_{d-1}(C_V^L) = Z_{d-1}(X)^L / \partial(X_d^L) \cong H^1(L;k),$$

naturally for subgroup restriction. Now $H^1(L;k) = \mathrm{Hom}_{\mathbb{F}_p}(L/\Phi(L), k)$. Restriction is dual to the linear map $L/\Phi(L) \to Q/\Phi(Q)$, whose kernel is $(L \cap \Phi(Q))/\Phi(L)$. Every linear functional on its image extends to $Q/\Phi(Q)$, so the displayed cokernel dimension follows. □

Theorem A.2 (Individual irreducible spheres)**.** *Let G be a finite p-group and V a nontrivial irreducible real G-representation. The following are equivalent:*

1. *C_V is res-onto;*
2. *$\Phi(G)$ acts trivially on V;*
3. *$G/\ker(V)$ is cyclic of order p.*

Consequently, a finite p-group is res-onto if and only if it is elementary abelian.

Proof. Put $Q = G/\ker(V)$ and $d = \dim_{\mathbb{R}} V$, and regard V as a faithful irreducible real Q-representation. If Q is not elementary abelian, choose a subgroup $L \leq \Phi(Q)$ of order p. In this case $d \geq 2$, since a faithful one-dimensional real representation has image of order at most two. Proposition A.1 gives a one-dimensional cokernel on restriction from Q to L in degree $d-1$. Inflation and the inverse image of L give the same failure for G. Thus res-onto implies Q elementary abelian.

A nontrivial faithful irreducible real representation of an elementary abelian p-group has image C_p: for $p = 2$ it is a sign character, and for odd p it is a rotation plane coming from a nontrivial complex character. Quotients carry $\Phi(G)$ onto $\Phi(Q)$, so conditions (2) and (3) are equivalent as well.

For $Q = C_p$, choose a generator g. A permutation homotopy model for the sphere is $k[C_2] \to k$ in degrees $1, 0$ when $p = 2$, and $k[C_p] \xrightarrow{g-1} k[C_p] \to k$ in degrees $2, 1, 0$ when p is odd; the maps to k are the augmentations. After inflation, a subgroup of G either has full image in C_p, in which case its invariant complex equals that for G, or has trivial image, in which case its only homology is the top fundamental class. The invariant norm cycle in the top term maps to that class. Restriction is surjective in every degree. This proves all three equivalences.

Finally all irreducible real representations kill $\Phi(G)$ if and only if $\Phi(G) = 1$: the regular representation is faithful and a sum of irreducibles. A finite p-group has trivial Frattini subgroup if and only if it is elementary abelian. Repeated canonical fixed-dimension labels cause no ambiguity, since the fixed-dimension function determines the kernel by $g \in \ker(V)$ if and only if $\dim V^{\langle g \rangle} = \dim V$. □

Example A.3 (A stable class that does not lift). Let $G = \langle g \rangle \cong C_{p^2}$, let $H = \langle g^p \rangle$, and put $e = g - 1$ and $A = kG \cong k[e]/(e^{p^2})$. Let $\epsilon : A \to k$ be the augmentation. The faithful rotation plane has sphere complex

$$C = (A \xrightarrow{e} A \xrightarrow{\epsilon} k) \qquad \text{in degrees } 2, 1, 0.$$

In particular C is itself a canonical irreducible sphere object. Since $g^p - 1 = e^p$, we have $A^H = e^{p^2-p}A$ and $A^G = ke^{p^2-1}$. Therefore

$$H_1(C^H) = e^{p^2-p}A/e^{p^2-p+1}A \cong k, \qquad H_1(C^G) = ke^{p^2-1}.$$

The first group is fixed by the residual G/H-action, since $g - 1 = e$ acts trivially on its quotient. Nevertheless restriction $H_1(C^G) \to H_1(C^H)$ is zero, because $p^2 - 1 \geq p^2 - p + 1$. Thus even a stable class in a single irreducible sphere twist need not lift without changing the twist.

This calculation shows that the stable-class lifting assertion in [20, Proposition 6.1] does not hold as stated. It also rules out the Dedekind-group characterization in [20, Theorem 7.3], since every subgroup of C_{p^2} is normal. These references are to the cited version 4; Theorem A.2 supplies the corrected classification. The elementary-abelian positive case is also recorded in [20, Example 7.2(a)], following the calculations of [3, Proposition 12.10].

Lifting conjugate products

For representation spheres, induction of representations corresponds to the topological norm; see [19, Remark 5.3], which credits this observation to Gallauer. Here we lift the conjugate product of a chosen section.

Proposition A.4 (Conjugate products of sphere sections)**.** *Let G be a finite p-group, let k have characteristic p, and let $H \trianglelefteq G$ have index n. Let V be an actual finite-dimensional real H-representation and let*

$$f \in \mathrm{Hom}_{\mathcal{K}(H)}(k[i], C_V).$$

There is a section

$$F \in \mathrm{Hom}_{\mathcal{K}(G)}(k[ni], C_{\mathrm{Ind}_H^G V})$$

whose restriction, under an H-equivariant sphere-chain identification

$$\mathrm{Res}_H^G C_{\mathrm{Ind}_H^G V} \simeq \bigotimes_{t \in G/H} {}^t C_V,$$

is $\bigotimes_{t\in G/H} {}^t f$. These identifications can be chosen to be the join and subdivision identifications in the proof.

Proof. Choose an H-admissible sphere triangulation K for V and an H-invariant cycle $z \in A(K)_i$ representing f. Choose and order left coset representatives $t_1, \dots, t_n$. Form the join L of the transported copies $t_j K$. If $g t_j = t_{\sigma_g(j)} h_{g,j}$, define the action of g on the jth factor by $h_{g,j}$ followed by transport to the $\sigma_g(j)$th factor. This defines a simplicial G-action on L, whose realization is the unit sphere of $\mathrm{Ind}_H^G V$.

The chain identification of a join with a tensor product sends

$$z_N = (t_1 z) \otimes \cdots \otimes (t_n z)$$

to a degree-ni cycle in $A(L)$. Since every internal $h_{g,j}$ fixes z, the action of g on this tensor contributes just $\mathrm{sgn}(\sigma_g)^i$. If p is odd, every cycle of the permutation σ_g has odd length, since its order is a power of p, so this sign is 1. If $p = 2$, every sign is 1 in k. Hence z_N is a G-invariant cycle.

The subdivision $\mathrm{sd}\, L$ is G-admissible. Naturality of the subdivision map therefore gives a G-invariant cycle $s(z_N)$ in the admissible sphere complex $A(\mathrm{sd}\, L)$, and this is the required global section F.

Normality of H means that H preserves every join factor. Since each factor is H-admissible, L is H-admissible. The subdivision lemma therefore makes s an H-chain homotopy equivalence. On restriction to H, its inverse identifies $s(z_N)$ with z_N, which represents the asserted tensor product of conjugate classes in the permutation homotopy category. If $V = 0$, use the empty complex and $A(\varnothing) = k$; the construction is the same scalar tensor product. □

Remark A.5 (Scope and choices). The proposition proves existence of a lift of a conjugate product. It does not assert that every stable individual section lifts, nor that the construction defines a canonical norm operation on homotopy classes. Changing a representative of f leaves its restricted product class unchanged, but this proof does not identify the resulting global lifts. Any two isomorphisms in $\mathcal{K}(G)$ from the constructed sphere object to a fixed canonical effective sphere object differ by multiplication by a nonzero scalar: invertibility identifies its endomorphism ring

with $\mathrm{End}_{\mathcal{K}(G)}(k) = k$. Such scalars do not affect homogeneous-prime membership and can be absorbed into the lifted section when necessary.

The induced representation is actual. Its nontrivial irreducible summands give nonnegative canonical twists; its trivial summands give an ordinary shift. Thus the global target belongs to the canonical effective ring, with no negative representation twists required.

Corollary A.6 (Powers of stable sections, with induced target)**.** *Let $H \trianglelefteq G$ have index n, let W be an actual real G-representation, and suppose*

$$f \in \mathrm{Hom}_{\mathcal{K}(H)}(k[i], \mathrm{Res}^G_H C_W)$$

is fixed by the conjugation action of G on this group. Then f^n lifts to a global section with target $C_{\mathrm{Ind}^G_H \mathrm{Res}^G_H W}$, under the H-identification of that target with $(\mathrm{Res}^G_H C_W)^{\otimes n}$.

Proof. The H-representation identification

$$\mathrm{Res}^G_H \mathrm{Ind}^G_H \mathrm{Res}^G_H W \cong (\mathrm{Res}^G_H W)^{\oplus n}$$

sends the tth transported copy of w to tw in the tth copy of W. Under it, the conjugate factors of the preceding proposition represent ${}^t f$. Their classes all equal f, so their tensor product is f^n. □

The lift belongs to the canonical effective ring of G. Its comparison with f^n uses an identification after restriction to H; it does not identify the global target with $C_W^{\otimes n}$. Indeed, $\mathrm{Ind}^G_H \mathrm{Res}^G_H W$ and $W^{\oplus n}$ can be different G-representations even though their restrictions to H agree.

Remark A.7 (Nonnormal subgroups). For any $H \leq G$, the same join construction and sign argument produce a global cycle in $C_{\mathrm{Ind}^G_H V}$. Its tensor-product restriction formula holds over every subgroup of the core $\mathrm{core}_G(H) = \bigcap_{g \in G} gHg^{-1}$, since such a subgroup preserves all join factors and the join is admissible for it. For nonnormal H, the join need not be H-admissible, so the subdivision map cannot be declared an H-homotopy equivalence by this argument. No general nonnormal restriction formula or homotopy-invariant norm functor is asserted. Lemma 8.3 avoids this issue entirely.

AI declaration

The author used ChatGPT as a tool for interactive proofreading, error detection, and improving the clarity of several arguments. ChatGPT also assisted with literature searches, proof development, and computational checks. The author is responsible for the mathematical content and references.

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Email address: hkshin@gmail.com