Products Preserve Riemann Integrability

A Local Oscillation Proof

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A proof about boundedness, local variation, and how Darboux estimates are built from pointwise identities.

1. Estimating a product difference

If f and g are Riemann integrable on a compact interval, then their product fg is Riemann integrable. We prove this by estimating the change in the product between two points of the same interval.

The proof below focuses on a local estimate. It asks:

If x and y lie in the same small interval, how much can $f(x)g(x)$ differ from $f(y)g(y)$?

The answer is encoded in the elementary identity

$$f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)).$$

The first term measures the change in f, weighted by a bounded value of g; the second measures the change in g, weighted by a bounded value of f.

Thus, on every interval I, the local oscillation of the product is controlled by the local oscillations of the factors:

$$\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g),$$

where M_f and M_g are global bounds for $|f|$ and $|g|$.

Once this local estimate is known, the Darboux criterion applies: choose a common refinement on which the oscillation sums for f and g are small, and the oscillation sum for fg will be small too.

The estimates follow these steps:

$$\text{expand a difference} \longrightarrow \text{isolate the moving pieces}$$
$$\longrightarrow \text{use boundedness} \longrightarrow \text{sum local estimates globally.}$$

The bounds hold regardless of the signs of f and g.

2. Darboux notation and oscillation

Let $u : [a, b] \to \mathbb{R}$ be bounded. If $I \subseteq [a, b]$ is a nonempty subinterval, define the oscillation of u on I by

$$\operatorname{osc}_I(u) = \sup_{t \in I} u(t) - \inf_{t \in I} u(t).$$

Let

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b,$$

be a partition of $[a, b]$. Write

$$I_i = [x_{i-1}, x_i], \qquad \Delta x_i = x_i - x_{i-1}.$$

Then

$$U(P, u) - L(P, u) = \sum_{i=1}^{n} \operatorname{osc}_{I_i}(u) \Delta x_i.$$

We shall use the Darboux criterion: a bounded function u is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P, u) - L(P, u) < \varepsilon.$$

The following lemma is a convenient way to estimate oscillations. It lets us replace upper and lower extrema by arbitrary pairs of points.

Lemma 1 (Oscillation as a two-point supremum)**.** *Let $u : I \to \mathbb{R}$ be bounded on a nonempty interval I. Then*

$$\operatorname{osc}_I(u) = \sup_{x,y \in I} |u(x) - u(y)|.$$

Proof. Let

$$M = \sup_{t \in I} u(t), \qquad m = \inf_{t \in I} u(t).$$

For every $x, y \in I$,

$$|u(x) - u(y)| \leq M - m,$$

and therefore

$$\sup_{x,y \in I} |u(x) - u(y)| \leq M - m.$$

Conversely, fix $\eta > 0$. By the defining properties of the supremum and infimum, there exist $x, y \in I$ such that

$$u(x) > M - \frac{\eta}{2}, \qquad u(y) < m + \frac{\eta}{2}.$$

Hence

$$|u(x) - u(y)| \geq u(x) - u(y) > M - m - \eta.$$

Since $\eta > 0$ was arbitrary,

$$\sup_{x,y \in I} |u(x) - u(y)| \geq M - m.$$

Combining the two inequalities gives

$$\operatorname{osc}_I(u) = \sup_{x,y \in I} |u(x) - u(y)|.$$

□

3. The local product estimate

The central estimate is purely local. It does not use integrability. It only uses boundedness and the product-difference identity.

Lemma 2 (Product oscillation estimate)**.** *Let $f, g : I \to \mathbb{R}$ be bounded functions on a nonempty interval I. Suppose that*

$$|f(t)| \le M_f, \qquad |g(t)| \le M_g$$

for every $t \in I$. Then

$$\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g).$$

Proof. Let $x, y \in I$. We write

$$\begin{aligned} f(x)g(x) - f(y)g(y) &= f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y) \\ &= (f(x) - f(y))g(x) + f(y)(g(x) - g(y)). \end{aligned}$$

Taking absolute values, and using the bounds on f and g, gives

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |f(x) - f(y)|\,|g(x)| + |f(y)|\,|g(x) - g(y)| \\ &\le M_g|f(x) - f(y)| + M_f|g(x) - g(y)|. \end{aligned}$$

Taking the supremum over all $x, y \in I$ and using the preceding lemma, we obtain

$$\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g).$$

□

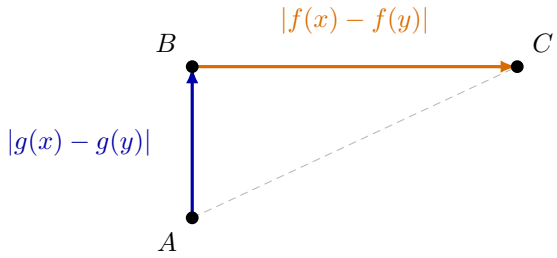


Figure 1: A schematic view of the product-difference identity. Here $A = (f(y), g(y))$, $B = (f(y), g(x))$, and $C = (f(x), g(x))$. Moving from A to C through B first changes the g-coordinate and then changes the f-coordinate. The vertical move is weighted by $|f(y)| \le M_f$, and the horizontal move is weighted by $|g(x)| \le M_g$.

4. Products preserve Riemann integrability

Theorem 1. *Let $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions. Then fg is Riemann integrable on $[a, b]$.*

Proof. Since Riemann integrable functions on compact intervals are bounded, choose constants $M_f, M_g > 0$ such that

$$|f(x)| \le M_f, \qquad |g(x)| \le M_g$$

for every $x \in [a, b]$.

Fix $\varepsilon > 0$. Choose

$$\rho = \frac{\varepsilon}{M_f + M_g + 1}.$$

Since f and g are Riemann integrable, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \rho$$

and

$$U(P_2, g) - L(P_2, g) < \rho.$$

Let

$$P = P_1 \cup P_2 = \{x_0, x_1, \ldots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b.$$

Because refinement decreases the difference between the upper and lower Darboux sums, we have

$$U(P, f) - L(P, f) < \rho, \qquad U(P, g) - L(P, g) < \rho.$$

For each i, write

$$I_i = [x_{i-1}, x_i], \qquad \Delta x_i = x_i - x_{i-1}.$$

By the product oscillation estimate,

$$\operatorname{osc}_{I_i}(fg) \le M_g \operatorname{osc}_{I_i}(f) + M_f \operatorname{osc}_{I_i}(g).$$

Multiplying by Δx_i and summing gives

$$\begin{aligned} U(P, fg) - L(P, fg) &= \sum_{i=1}^{n} \operatorname{osc}_{I_i}(fg)\Delta x_i \\ &\le M_g \sum_{i=1}^{n} \operatorname{osc}_{I_i}(f)\Delta x_i + M_f \sum_{i=1}^{n} \operatorname{osc}_{I_i}(g)\Delta x_i \\ &= M_g\big(U(P, f) - L(P, f)\big) + M_f\big(U(P, g) - L(P, g)\big) \\ &< M_g\rho + M_f\rho \\ &= (M_f + M_g)\rho \\ &< \varepsilon. \end{aligned}$$

Therefore, for every $\varepsilon > 0$, there exists a partition P such that

$$U(P, fg) - L(P, fg) < \varepsilon.$$

By the Darboux criterion, fg is Riemann integrable on $[a, b]$. □

5. From local oscillation to Darboux sums

The entire proof is organized around the estimate

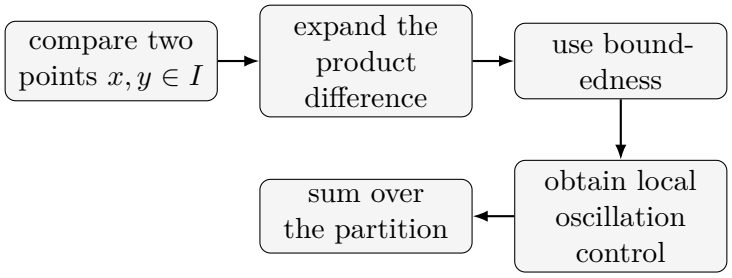
$$\boxed{\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g)}$$

for every nonempty subinterval $I \subseteq [a, b]$.

This estimate converts local information about f and g into local information about fg. Once we multiply by interval lengths and sum, the local estimate becomes a Darboux estimate:

$$U(P, fg) - L(P, fg) \le M_g\big(U(P, f) - L(P, f)\big) + M_f\big(U(P, g) - L(P, g)\big).$$

The argument proceeds as follows:



6. Why this avoids sign-splitting

Products of upper and lower bounds require attention to signs. If

$$\underline{f} \le f \le \overline{f}, \qquad \underline{g} \le g \le \overline{g},$$

it is not generally true that

$$\underline{f}\,\underline{g} \le fg \le \overline{f}\,\overline{g}.$$

Multiplication is monotone in each variable only under sign restrictions. This is why one often splits into positive and negative parts before multiplying minorants and majorants.

The oscillation proof bypasses that issue. It does not ask whether products of minorants minorize or products of majorants majorize. It asks instead:

How far apart can two values of the product be on the same small interval?

This question is answered directly by

$$f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)).$$

No positivity is required. The possible sign changes of f and g are absorbed by absolute values and boundedness.

7. Other Lipschitz constructions

Let $f_1, \dots, f_m : [a, b] \to \mathbb{R}$ be Riemann integrable, and suppose $F : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz on a bounded set containing their joint range. Put $h(x) = F(f_1(x), \dots, f_m(x))$. The function h is bounded, and for some constant $C \geq 0$,

$$\begin{aligned} &|F(f_1(x), \dots, f_m(x)) - F(f_1(y), \dots, f_m(y))| \\ &\quad \leq C \left(|f_1(x) - f_1(y)| + \cdots + |f_m(x) - f_m(y)| \right) \end{aligned}$$

Taking suprema over x, y in a nonempty subinterval I gives

$$\operatorname{osc}_I(h) \leq C \sum_{j=1}^{m} \operatorname{osc}_I(f_j).$$

Multiplying by interval lengths and summing gives the corresponding bound for the Darboux gaps. A common refinement makes the gaps for all f_j simultaneously small, proving that h is Riemann integrable.

For multiplication, $F(s, t) = st$ is not globally Lipschitz on all of $\mathbb{R}^2$, but it is Lipschitz on every bounded rectangle. Since Riemann integrable functions on compact intervals are bounded, this is enough.

For maximum, $F(s, t) = \max(s, t)$ is globally Lipschitz, and the same estimate gives

$$\operatorname{osc}_I(\max(f, g)) \leq \operatorname{osc}_I(f) + \operatorname{osc}_I(g).$$