What a Boundary Map Can Prove

Incidence, cancellation, gluing, and elimination

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Abstract

An edge diagram becomes a calculation when its local operations are exposed. Incidence and Cauchy–Binet count trees; an inverse solves an energy problem; coefficient extraction gives joint probabilities. Smith operations and higher boundary minors retain integral torsion. The later arguments perform further finite procedures: cancel the coloring sum by an edge toggle, diagonalize a small intersection form, remove tree transport labels by gauge changes, match the two branches of a node, and eliminate variables in a graph integral. Each procedure proves its own bounded conclusion. Separate reference sections retain the deeper Hodge, motivic, and global period theorems, with their imported inputs explicit; those frameworks are not prerequisites for the completed core calculations.

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1 The starting complex

Assign a number to each oriented edge. Can those numbers be differences of values at its endpoints? On a tree, choose one vertex value and propagate it along the unique paths. An extra edge introduces a check: the sum around its circuit must vanish. The useful change is to treat all these local differences as one linear operation. We will then ask which other questions become calculations with that operation.

Let $G = (V, E)$ be a finite connected undirected graph, with $n \geq 1$ vertices and m edges. Loops and parallel edges are allowed here; a loop is a cycle, and two parallel edges form a cycle of length two. Choose edge orientations and a field k. The cochain complex is

$$C^0(G;k) = k^V \xrightarrow{d} C^1(G;k) = k^E, \qquad (df)(e) = f(t(e)) - f(s(e)).$$

Its kernel consists of constants, since zero differences along a path force equal vertex values. There are no higher cochains, so rank–nullity gives

$$\dim H^1(G;k) = \dim(k^E/\operatorname{im} d) = m - n + 1. \tag{1}$$

Lemma 1.1 (The tree criterion)**.** *If $m = n-1$, then G is a tree. Equivalently, every edge assignment integrates to a vertex potential, unique up to a constant.*

Proof. The dimension formula makes d surjective. A cycle would define a nonzero functional on edge assignments by summing around it, with traversal signs. This functional annihilates df by telescoping, contradicting surjectivity. Thus G has no cycles. Conversely, on a tree, integrate edge values along the unique paths from a root; this gives a potential, unique up to a constant. The argument works in every characteristic, including two. For $n = 1, m = 0$, it gives the single vertex. □

For the path $1 \to 2 \to 3$, edge values a, b integrate to $(0, a, a + b)$. Adding an edge $3 \to 1$ with value c introduces the obstruction $a + b + c$. We now keep this complex and ask questions it can answer beyond recognizing a tree.

Here is a first calculation before any further framework is introduced. Orient a triangle as $1 \to 2 \to 3 \to 1$, and let a column record the head minus the tail of an edge. Its incidence matrix is

$$B = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Fix the value at vertex 3 to be zero. Removing that row gives D, and multiplying the remaining differences back to vertices gives

$$D = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad DD^{\mathsf{T}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The determinant is 3, the number of spanning trees. Solving

$$DD^{\mathsf{T}} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

gives $(p_1, p_2) = (2/3, 1/3)$ and the effective resistance $2/3$ between vertices 1 and 3. These are not coincidences to be taken on trust: Section 3 proves both readings of the matrix, then derives the

probabilities of selecting specified edges in a random spanning tree. A determinant, a linear solve, and a coefficient extraction answer three different questions about the same graph.

The finite route through Sections 1, 3, and 4 is complete using linear algebra and integer operations. It does not require Section 2. That section answers a different question: how can we replace scalar vertex values by varying coefficient spaces while retaining solutions and failures of gluing? Its explicit constructions are also a second account of the determinant identities.

The later routes begin when the question changes. For higher cells, a boundary basis may have a nontrivial integer index. For colorings, the coefficients obey inequalities not supplied by the tree projection; a new ring and a deep positivity theorem enter. For nodal curves, a graph omits the attachment points that can change the symmetry group. For period integrals, the cycle determinant becomes a polynomial with a singular zero set. Each extension earns its additional objects by an operation the earlier data cannot perform.

2 A graph as a derived gluing problem

Suppose vertex values now lie in different vector spaces, and each endpoint has its own map into the space attached to an edge. A single scalar incidence matrix no longer describes the problem. We construct its replacement from those local maps and verify it cell by cell. The resulting two-term complex retains both compatible choices and the equations they fail to satisfy. The language of derived global sections records this construction and its behaviour under changes of coefficients; it is not needed for the scalar calculations of Section 3.

2.1 A projective resolution computes the incidence complex

Form the incidence category of G: its objects are vertices and edges, and each endpoint incidence gives an arrow $v \to e$. A loop has two such arrows from its vertex. A cellular sheaf $\mathcal{F}$ assigns vector spaces $\mathcal{F}_v, \mathcal{F}_e$ and linear maps $\rho_{v,e} : \mathcal{F}_v \to \mathcal{F}_e$ to these arrows. Its global sections are compatible choices of local values. An orientation records compatibility as

$$\Gamma(G, \mathcal{F}) = \ker\left(\bigoplus_v \mathcal{F}_v \xrightarrow{d_{\mathcal{F}}} \bigoplus_e \mathcal{F}_e \right), \qquad (d_{\mathcal{F}} s)_e = \rho_{t(e),e} s_{t(e)} - \rho_{s(e),e} s_{s(e)}.$$

For a loop the two restriction maps are distinguished. The constant sheaf $\underline{k}$ gives the original coboundary. A local system is the case in which all restriction maps are isomorphisms; it retains transport along paths.

The category $\mathcal{A}_G$ of these diagrams is abelian: kernels and cokernels are taken at each cell. Global sections are left exact, since taking compatible values commutes with kernels. A surjection of diagrams need not give a surjection of compatible values. Its missing information is precisely what the derived functor records.

For a cell σ, let P_σ be the diagram whose value at τ is the vector space freely generated by arrows $\sigma \to \tau$. Then

$$\mathrm{Hom}_{\mathcal{A}_G}(P_\sigma, \mathcal{F}) = \mathcal{F}_\sigma.$$

Evaluation is exact, so P_σ is projective. There is a projective resolution

$$0 \longrightarrow \bigoplus_e P_e \longrightarrow \bigoplus_v P_v \longrightarrow \underline{k} \longrightarrow 0. \tag{2}$$

At a vertex, the last nonzero map is $k \to k$, the identity. At an edge, it is $k^2 \to k$, addition; its kernel is the difference of the endpoint generators. This verifies exactness cell by cell, including loops. Applying $\mathrm{Hom}(-, \mathcal{F})$ computes the derived functor:

$$\mathrm{R}\Gamma(G, \mathcal{F}) = \mathrm{RHom}_{\mathcal{A}_G}(\underline{k}, \mathcal{F}) \simeq \left[\bigoplus_v \mathcal{F}_v \xrightarrow{d_{\mathcal{F}}} \bigoplus_e \mathcal{F}_e \right]. \tag{3}$$

The bracketed complex has degrees zero and one. The usual incidence formula has appeared as a derived-functor calculation, valid for arbitrary coefficient diagrams. Cellular sheaf cohomology and its derived constructions are developed in [27].

We work in the derived category, where a quasi-isomorphism—a map inducing isomorphisms on every cohomology group—is invertible. Thus a resolution can be replaced by the object it resolves, while its functorial maps remain available. A short exact sequence of complexes gives an exact triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

Here $A[1]^i = A^{i+1}$, with the usual change of sign in the differential. Taking cohomology turns the triangle into a long exact sequence. Keeping the final arrow is useful: it records the obstruction to lifting a class, rather than only the dimensions of the groups.

2.2 Removing an edge produces an obstruction map

Let $H = G \setminus e$, retaining every vertex, and restrict $\mathcal{F}$ to H. Dropping the edge component is a surjection of the two-term complexes in (3); its kernel is $\mathcal{F}_e$ in degree one. Hence there is an exact triangle

$$\mathcal{F}_e[-1] \longrightarrow \mathrm{R}\Gamma(G, \mathcal{F}) \longrightarrow \mathrm{R}\Gamma(H, \mathcal{F}|_H) \xrightarrow{\delta_e} \mathcal{F}_e. \tag{4}$$

Its cohomology sequence is

$$\begin{aligned} 0 \longrightarrow H^0(G, \mathcal{F}) \longrightarrow H^0(H, \mathcal{F}) \xrightarrow{\delta_e} \mathcal{F}_e \\ \longrightarrow H^1(G, \mathcal{F}) \longrightarrow H^1(H, \mathcal{F}) \longrightarrow 0. \end{aligned}$$

The connecting map evaluates the discrepancy between the two endpoint values. All effects of the extra edge on global gluing are concentrated in this one morphism.

For constant coefficients, an edge joining two different components of H makes δ_e surjective: their two independent constants can have any prescribed difference. It removes one component and introduces no first cohomology. If the endpoints already lie in the same component, $\delta_e = 0$ and the edge introduces one first-cohomology class. Components and cycles are the two outcomes of the same exact triangle.

For a local system, contracting a nonloop edge transports its endpoint fibers into one fiber. This gives a quasi-isomorphism on derived global sections. One way to see it is to choose that transport as an identification: the difference of the two endpoint values and the edge value form an acyclic two-term summand, which can be cancelled. Contracting a spanning tree therefore leaves one fiber W and $r = m - n + 1$ loop transports $T_1, \dots, T_r$.

Proposition 2.1 (Derived gluing with holonomy)**.** *For a rank-a local system on a connected graph,*

$$\mathrm{R}\Gamma(G, \mathcal{L}) \simeq \left[W \xrightarrow{\Delta} W^{\oplus r} \right], \qquad \Delta(w) = ((T_i - I)w)_{i=1}^r.$$

Consequently

$$H^0(G,\mathcal{L}) = \bigcap_i \ker(T_i - I), \qquad \dim H^1(G,\mathcal{L}) = (r-1)a + \dim H^0(G,\mathcal{L}),$$

and $H^j(G,\mathcal{L}) = 0$ *for* $j \geq 2$. *On a tree, evaluation at any chosen root gives* $\mathrm{R}\Gamma(G,\mathcal{L}) \simeq W[0]$.

Proof. The preceding acyclic cancellations identify the complex with the displayed loop complex. Its kernel is the common invariant subspace, and its Euler characteristic is $a - ra$. The projective resolution (2) has length one, so there are no higher derived global sections. For $r = 0$ the remaining complex is just W in degree zero. □

Example 2.2 (A cycle can disappear in a realization)**.** Take a rank-one local system on a triangle with total holonomy $\lambda \in k^\times$. Removing one edge leaves a tree, so (4) reduces to the two-term complex

$$[k \xrightarrow{\lambda - 1} k].$$

For $\lambda = 1$ both cohomology groups are k. For $\lambda \neq 1$ the complex is acyclic. The underlying cycle persists, while the chosen transport makes its gluing discrepancy invertible. The twisted determinant in Section 3 will measure exactly this change.

We now keep integral coefficients. A rational quasi-isomorphism can erase finite gluing obstructions; derived duality and the determinant functor recover their arithmetic contribution.

2.3 Determinant lines turn exactness into arithmetic

An exact triangle describes how complexes fit together. A determinant functor turns that gluing into a multiplication of one-dimensional lines. Over $\mathbb{Z}$, these lines also measure the finite groups that disappear after passage to $\mathbb{Q}$.

For a bounded complex $P^\bullet$ of finite free abelian groups, put

$$\det P^\bullet = \bigotimes_i \left(\bigwedge^{\operatorname{rank} P^i} P^i\right)^{(-1)^i}.$$

An exponent -1 means the dual line. The determinant construction passes to perfect complexes and sends an exact triangle $P' \to P \to P'' \to P'[1]$ to

$$\det P \cong \det P' \otimes \det P''.$$

The usual parity convention handles orientation signs; we will take absolute values. This is the determinant functor of Knudsen–Mumford [28]; its two-term construction is given explicitly in [29].

If $P \otimes \mathbb{Q}$ is acyclic, cancellation of boundaries gives a canonical rational trivialization $\phi_P : \det P \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$. For an integral generator ℓ_P, define

$$\mathfrak{t}(P) = |\phi_P(\ell_P)|^{-1}.$$

This is the absolute Reidemeister torsion with the convention that $[\mathbb{Z} \xrightarrow{q} \mathbb{Z}]$, in degrees $-1, 0$, has torsion $|q|$.

Proposition 2.3 (Arithmetic torsion and exact triangles)**.** *If all cohomology groups of P are finite, then*

$$\mathfrak{t}(P) = \prod_i |H^i(P)|^{(-1)^i}. \tag{5}$$

For an exact triangle of such complexes, $\mathfrak{t}(P) = \mathfrak{t}(P')\mathfrak{t}(P'')$.

Proof. Write $Z^i = \ker d^i$ and $B^i = \operatorname{im} d^{i-1}$. The sequence $0 \to Z^i \to P^i \to B^{i+1} \to 0$ splits over $\mathbb{Z}$, since B^{i+1} is free. The inclusion $B^i \subset Z^i$ has index $|H^i(P)|$. In the determinant line, the boundary factors in adjacent degrees cancel. Replacing an integral volume of Z^i by one of B^i contributes $|H^i(P)|^{(-1)^{i+1}}$ to $\phi_P(\ell_P)$. This proves the formula. Determinant multiplicativity and the compatible rational trivializations prove the assertion for triangles. Equivalently, the alternating product of orders in their finite long exact cohomology sequence is one. □

Here is the dual information carried by a finite group. Let $A : U \to L$ be a map of free abelian groups of equal finite rank r, invertible over $\mathbb{Q}$, and put $F = L/AU$. The complex $K = [U \xrightarrow{A} L]$, in degrees $-1, 0$, is a free resolution of F. Its determinant section $\delta_A = \bigwedge^r A$ belongs to $\det K$, and $\phi_K(\delta_A) = 1$. Its coefficient in an integral generator therefore has absolute value $\mathfrak{t}(K) = |F|$. The shifted derived dual is represented by

$$R\operatorname{Hom}_{\mathbb{Z}}(K, \mathbb{Z})[1] \simeq [L^* \xrightarrow{A^*} U^*],$$

where a change of differential sign gives the displayed representative. The dual has its sole cohomology in degree zero:

$$\operatorname{coker} A^* \cong \operatorname{Ext}^1_{\mathbb{Z}}(F, \mathbb{Z}) \cong F^\vee := \operatorname{Hom}(F, \mathbb{Q}/\mathbb{Z}). \tag{6}$$

Indeed, applying $R\operatorname{Hom}(F, -)$ to the coefficient triangle $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}[1]$ gives the last isomorphism: $\mathbb{Q}$ is injective and has no nonzero homomorphisms from F. This is the finite part of the universal coefficient theorem [31].

The resulting pairing is concrete and canonical:

$$(L/AU) \times (U^*/A^*L^*) \longrightarrow \mathbb{Q}/\mathbb{Z}, \qquad ([x], [y]) \longmapsto y(A_{\mathbb{Q}}^{-1}x) \pmod{\mathbb{Z}}.$$

Changing either lift changes the value by an integer. If it vanishes against every y, then $A_{\mathbb{Q}}^{-1}x \in U$, so $[x] = 0$; the same argument on the other side proves perfection. Thus $|F^\vee| = |F|$, and the determinant contributions of the resolution and its shifted dual multiply to

$$\mathfrak{t}(K)\,\mathfrak{t}\big(R\operatorname{Hom}_{\mathbb{Z}}(K, \mathbb{Z})[1]\big) = |F|\,|F^\vee| = |F|^2. \tag{7}$$

This arithmetic square will be the weight of a higher tree.

The integral lattice and the cell coordinates remain part of the enumeration. For example, the reduced integral complexes of C_3 and C_4 are quasi-isomorphic: each has a single $\mathbb{Z}$ in degree -1. Their tree counts are 3 and 4. The determinant functor organizes exactness and torsion; the metric and the specified cells supply the volumes and the choices being counted. Multiplicativity of Reidemeister torsion and its determinant-line interpretation are developed in [30].

2.4 An exact triangle proves arithmetic deletion–contraction

Let L be free abelian of rank r, and let $A : \mathbb{Z}^E \to L$ have full rational rank r. For each r-element set S whose columns span $L \otimes \mathbb{Q}$, put $F_S = L/A\mathbb{Z}^S$, and define

$$\mathcal{P}_L(A; w) = \sum_S |F_S|^2 \prod_{s \in S} w_s.$$

The sum is zero when no such set exists, and is one in rank zero. Choose a nonzero column $a_e = qu$, where $u \in L$ is primitive and $q > 0$. Write $\bar{L} = L/\mathbb{Z}u$, and let $\bar{A}$ be the remaining columns in this free quotient. This is contraction by the saturated rank-one lattice generated by a_e.

Theorem 2.4 (Arithmetic deletion–contraction)**.** *With these integral normalizations,*

$$\mathcal{P}_L(A; w) = \mathcal{P}_L(A \setminus e; w) + q^2 w_e \, \mathcal{P}_{\bar{L}}(\bar{A}; w). \tag{8}$$

For a zero column only the deletion term occurs.

Proof. Separate the sets according to whether they contain e. For $S = \{e\} \cup R$, the full-rank condition is precisely that the columns indexed by R form a rational basis of $\bar{L}$. There is a short exact sequence of two-term free complexes

$$0 \longrightarrow [\mathbb{Z}e \xrightarrow{q} \mathbb{Z}u] \longrightarrow [\mathbb{Z}^S \xrightarrow{A_S} L] \longrightarrow [\mathbb{Z}^R \xrightarrow{\bar{A}_R} \bar{L}] \longrightarrow 0.$$

It gives an exact triangle of perfect complexes. All three are rationally acyclic, so Proposition 2.3 gives $|F_S| = q|\bar{L}/\bar{A}\mathbb{Z}^R|$. Its finite cohomology sequence is the extension

$$0 \longrightarrow \mathbb{Z}/q\mathbb{Z} \longrightarrow F_S \longrightarrow \bar{L}/\bar{A}\mathbb{Z}^R \longrightarrow 0.$$

The shifted dual contributes the same factor q. Multiplying the two torsions, and then summing the monomials, proves the recurrence. A zero column cannot belong to a rational basis. □

For graphs take $L = \mathbb{Z}_0^V$, the zero-sum vertex lattice. Every non-loop incidence column is primitive, so $q = 1$; identifying its endpoints identifies $\bar{L}$ with the zero-sum lattice of G/e. Integral path integration gives $F_S = 0$ for a tree. Thus the theorem gives

$$\tau(G; w) = \tau(G \setminus e; w) + w_e \tau(G/e; w),$$

with value zero for a disconnected deletion. The exact triangle explains why contraction has coefficient one in graphs and an arithmetic coefficient in higher dimension.

For instance, attach two 2-cells to a circle by maps of degrees 2 and 3. The top boundary is $[2 \;\; 3]$. Its two rational trees have first homology $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, with duals in top cohomology. Their polynomial is $4w_1 + 9w_2$. Contracting the first column leaves a rank-zero quotient and gives $9w_2 + 2^2 w_1$; the extra factor records the finite cohomology of the rank-one complex being removed.

2.5 The determinant line carries a probability law

The preceding arithmetic coefficients become probabilities through the metric on the determinant line. This operation applies to any subspace with a specified orthonormal coordinate system. Its graph and cellular instances will follow by choosing a boundary map.

Let $V \subset \mathbb{R}^E$ have dimension r. Its determinant line $\det V = \bigwedge^r V$ sits inside $\bigwedge^r \mathbb{R}^E$. Choose a unit generator

$$\xi \in \det V, \qquad \xi = \sum_{|T|=r} \xi_T e_T, \qquad \sum_T \xi_T^2 = 1,$$

where e_T is the ordered wedge of coordinate vectors. Orientation changes signs, while ξ_T^2 is unchanged. Give an r-element set T probability ξ_T^2, and let P be orthogonal projection onto V.

Theorem 2.5 (Exterior-power inclusion law)**.** *For every coordinate set $S \subseteq E$,*

$$\Pr(S \subseteq T) = \|\iota_{e_S}\xi\|^2 = \det\big(\langle Pe_i, Pe_j\rangle\big)_{i,j\in S} = \det P_S.$$

Here ι_{e_S} is successive interior multiplication by the vectors indexed by S, in a fixed order. Sets larger than r have zero probability, and the empty-set determinant is one.

Proof. In the coordinate expansion of ξ, contraction by e_S keeps precisely those terms whose index contains S. The remaining coordinate wedges are distinct and orthonormal, so its squared norm is the sum of ξ_T^2 over these terms.

Since ξ lies in $\bigwedge^r V$, contraction by e_i equals contraction by Pe_i. The metric and the unit volume ξ give the Hodge-star isometry on V. Successive contraction of its top volume is, up to sign, the Hodge star of the wedge of the contracted vectors. Thus

$$\|\iota_{e_S}\xi\|^2 = \left\|\bigwedge_{i\in S} Pe_i\right\|^2 = \det\big(\langle Pe_i, Pe_j\rangle\big)_{i,j\in S}.$$

The last equality is the exterior metric: a wedge's squared volume is its Gram determinant. Because P is self-adjoint and idempotent, $\langle Pe_i, Pe_j\rangle = \langle e_i, Pe_j\rangle$. These are the entries of P_S. □

Let A be an integral $r \times |E|$ boundary matrix of full row rank, with weights $w_e > 0$, and put $Y = A\,\mathrm{diag}(\sqrt{w_e})$. Take $V = \mathrm{im}\, Y^{\mathsf{T}}$. Pulling an integral volume through $\bigwedge^r Y^{\mathsf{T}}$ and normalizing gives

$$\xi_T^2 = \frac{(\det A_T)^2 \prod_{e\in T} w_e}{\det(YY^{\mathsf{T}})}, \qquad P = Y^{\mathsf{T}}(YY^{\mathsf{T}})^{-1}Y. \tag{9}$$

The integral resolution identifies $|\det A_T|$ with the order of the finite homology group; its derived dual identifies the matching cohomology factor. The inclusion law is then an exterior contraction of that same determinant line. Torsion weights and random-tree correlations arise from the arithmetic and metric realizations of one boundary presentation.

3 The same boundary map counts, conducts, and correlates

The coboundary records components in its kernel and cycles in its cokernel. With an inner product, it also gives a Laplacian: its determinants count trees, its inverse measures electrical resistance, and its orthogonal projection gives random-tree correlations.

Throughout this section, $G = (V, E)$ is a finite connected undirected loopless graph, possibly with parallel edges. Put $N = |V|$ and $M = |E|$, orient each edge arbitrarily, and let

$$B : \mathbb{R}^E \longrightarrow \mathbb{R}^V, \qquad Be = \mathbf{e}_{t(e)} - \mathbf{e}_{s(e)}$$

be its incidence matrix. Here $\mathbf{e}_v$ is the coordinate vector at v. Give each edge a positive conductance c_e, write $W = \operatorname{diag}(c_e)$, and set

$$A = BW^{1/2}, \qquad d_c = A^{\mathsf{T}}, \qquad L = AA^{\mathsf{T}} = BWB^{\mathsf{T}}.$$

The operator d_c is the usual coboundary after rescaling edge coordinates: $(d_c f)_e = \sqrt{c_e}\big(f(t(e)) - f(s(e))\big)$. The weights change the geometry of the spaces, while the cohomology remains that of the graph. Except where explicitly stated, all conductances in the examples are 1.

3.1 A finite Hodge theorem

For every vertex function f,

$$f^{\mathsf{T}} L f = \|d_c f\|^2 = \sum_{e \in E} c_e \big(f(t(e)) - f(s(e))\big)^2. \tag{10}$$

Consequently L is positive semidefinite and, by connectedness, its kernel consists exactly of constants. Let L^+ act as the inverse of L on the orthogonal complement $\mathbf{1}^\perp$, and as zero on constants. This is its Moore–Penrose pseudoinverse; no choice of root enters its definition.

Proposition 3.1 (Hodge decomposition and projection)**.** *There is an orthogonal decomposition*

$$\mathbb{R}^E = \operatorname{im} d_c \;\oplus\; \ker A.$$

The second summand has dimension $M - N + 1$ and represents $H^1(G;\mathbb{R})$ in the rescaled coordinates. The orthogonal projection onto the first summand is

$$K = A^{\mathsf{T}} L^+ A. \tag{11}$$

Proof. The identity $(\operatorname{im} A^{\mathsf{T}})^\perp = \ker A$ gives the orthogonal decomposition. Connectedness gives $\operatorname{rank} A = N-1$, hence $\dim \ker A = M-N+1$. To compute the projection, observe that $K = A^{\mathsf{T}} L^+ A$ is symmetric and

$$K^2 = A^{\mathsf{T}} L^+ L L^+ A = K.$$

It vanishes on $\ker A$. For a vector $A^{\mathsf{T}} x$ in the other summand,

$$K A^{\mathsf{T}} x = A^{\mathsf{T}} L^+ L x = A^{\mathsf{T}}(x - \Pi_0 x) = A^{\mathsf{T}} x, \qquad \Pi_0 = \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathsf{T}},$$

because A^{T} kills constants. These two properties identify K as the required projection.

There is also a description in the complex of Section 2. Put

$$h = L^+ A : C^1 \longrightarrow C^0, \qquad \Pi_0 = \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathsf{T}}, \qquad \Pi_1 = I - A^{\mathsf{T}} L^+ A.$$

The spectral definition of L^+ gives

$$h d_c = I - \Pi_0, \qquad d_c h = I - \Pi_1.$$

Thus $dh + hd = I - \Pi$ on the complex. Moreover $A\Pi_1 = 0$, and Π_1 acts as the identity on $\ker A$. It is self-adjoint, so it is the orthogonal projection onto that kernel. The complex deformation retracts onto its harmonic cohomology, with zero differential: constants in degree zero and $\ker A$ in degree one. Its complementary degree-one projection is $K = d_c h$, and the Euler characteristic gives the stated dimension. The orthogonal decomposition is the metric realization of this deformation retract. □

There are no 2-cochains here. Thus a harmonic 1-cochain means a cochain annihilated by the adjoint of d_c; the corresponding edge Laplacian is $d_c d_c^{\mathsf{T}}$. If G is a tree, the harmonic summand vanishes and $K = I_M$. Every edge assignment is a gradient. If G is a coherently oriented triangle with conductances a, b, c, its harmonic line is generated by

$$z = (a^{-1/2}, b^{-1/2}, c^{-1/2})^{\mathsf{T}}, \qquad K = I_3 - \frac{zz^{\mathsf{T}}}{a^{-1} + b^{-1} + c^{-1}}.$$

The obstruction to integration is literally one line in a three-dimensional space.

Cohomology supplies the obstruction space; Hodge theory uses the inner product to choose harmonic representatives and a projection. This projection will give the tree probabilities.

3.2 A determinant that lists all spanning trees

Define the weighted tree polynomial

$$\tau(G; c) = \sum_{T \text{ spanning tree of } G} \prod_{e \in T} c_e.$$

For a one-vertex graph we use $\tau = 1$, the contribution of the empty tree. Choose a root r, delete its row from B to obtain $\widehat{B}$, and delete its row and column from L to obtain

$$\widehat{L} = \widehat{B} W \widehat{B}^{\mathsf{T}}.$$

Theorem 3.2 (Weighted matrix-tree theorem)**.** *For every root,*

$$\tau(G; c) = \det \widehat{L}. \tag{12}$$

If $\lambda_1, \dots, \lambda_{N-1}$ are the nonzero eigenvalues of L, then

$$\tau(G; c) = \frac{1}{N} \prod_{j=1}^{N-1} \lambda_j. \tag{13}$$

Proof. Set $Y = \widehat{B} W^{1/2}$. Cauchy–Binet expands the determinant into a sum over $(N-1)$-element sets of edge columns:

$$\det \widehat{L} = \det(YY^{\mathsf{T}}) = \sum_{|S| = N-1} (\det \widehat{B}_S)^2 \prod_{e \in S} c_e.$$

We identify the minors rather than evaluating the determinant blindly. If S contains a cycle, its signed incidence columns sum to zero around that cycle, so its determinant is zero. A cycle-free set of $N-1$ edges on N vertices is connected, hence is a spanning tree. For a tree, choose a leaf different from the root. Its row in $\widehat{B}_S$ has one nonzero entry, equal to 1 or -1. Expanding along that row removes the leaf and its edge. Repeating ends with the empty determinant 1, so every tree minor is 1 or -1. The surviving summands are exactly the tree weights, proving the first assertion.

The Hodge splitting of $\mathbb{R}^V$ into constants and their orthogonal complement gives

$$\operatorname{adj}(L) = \left(\prod_{j=1}^{N-1} \lambda_j \right) \Pi_0, \qquad \Pi_0 = \frac{1}{N} \mathbf{1}\mathbf{1}^{\mathsf{T}}.$$

Its diagonal entries are the grounded determinants just computed. This gives the spectral formula. For $N = 1$, the determinant line and the empty tree have volume one, as do the empty determinant and product. □

The determinant is useful because its expansion has been identified term by term with the desired objects. Section 2.5 gives an exterior-volume interpretation of the same sum, and cellular versions are discussed in [4]. The following examples reduce the determinant further by choosing subspaces on which the Laplacian has a simple action.

3.3 Symmetry turns eigenvalues into concrete counts

Example 3.3 (Complete and complete bipartite graphs)**.** For K_N, $L = NI - J$, where J is the all-ones matrix. Its nonzero eigenvalue is N with multiplicity $N-1$, so for $N \geq 2$

$$\tau(K_N) = N^{N-2}.$$

In particular K_4 has 16 trees and K_5 has 125.

For $K_{p,q}$ with $p, q \geq 1$, the vectors supported on the p-vertex part and summing to zero have eigenvalue q, with multiplicity $p-1$. The analogous subspace on the other part has eigenvalue p, with multiplicity $q-1$. On functions constant on each part, the remaining eigenvalues are 0 and $p+q$. Hence

$$\tau(K_{p,q}) = p^{q-1}q^{p-1}.$$

For instance $K_{2,3}$ has 12 trees, $K_{3,3}$ has 81, and the star $K_{1,q}$ has exactly one. All these counts follow from four invariant subspaces rather than a list of trees.

Example 3.4 (A weighted family behind Cayley's formula)**.** Choose positive numbers $x_1, \ldots, x_N$ and assign conductance $x_i x_j$ to the edge ij of K_N. If $S = \sum_i x_i$ and $x = (x_i)$, then

$$L = S\,\mathrm{diag}(x_i) - xx^{\mathsf{T}}.$$

Deleting vertex r and applying the rank-one determinant identity to its diagonal part gives, for $N \geq 2$,

$$\begin{aligned}\det \widehat{L} &= S^{N-1} \prod_{i \neq r} x_i \left(1 - \frac{\sum_{i \neq r} x_i}{S}\right) \\ &= \left(\prod_i x_i\right) S^{N-2}.\end{aligned}$$

Thus an entire polynomial identity sits above the numerical count:

$$\sum_T \prod_i x_i^{\deg_T(i)} = \left(\prod_i x_i\right)(x_1 + \cdots + x_N)^{N-2}.$$

For $N = 3$ and $(x_1, x_2, x_3) = (1, 2, 3)$, the edge conductances are $2, 3, 6$. The three tree weights are $6, 12, 18$, totaling 36, exactly $1 \cdot 2 \cdot 3 \cdot 6$. Setting every x_i to 1 projects this weighted identity to Cayley's formula; keeping the variables preserves degree information.

Example 3.5 (The hypercube and its characters)**.** For $d \geq 1$, the d-cube Q_d has vertex set $(\mathbb{Z}/2)^d$ and joins vertices differing in one coordinate. For $S \subseteq \{1, \ldots, d\}$, the character

$$\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$$

is an eigenfunction: flipping a coordinate in S changes its sign, and flipping any other coordinate leaves it fixed. Therefore

$$L\chi_S = 2|S|\chi_S.$$

These 2^d characters form an orthogonal basis, giving

$$\tau(Q_d) = \frac{1}{2^d}\prod_{k=1}^{d}(2k)^{\binom{d}{k}}.$$

Thus Q_2 has 4 trees, the ordinary cube Q_3 has

$$\frac{2^3\,4^3\,6}{8} = 384,$$

and Q_4 has

$$\frac{2^4\,4^6\,6^4\,8}{16} = 42\,467\,328.$$

There is a direct group-theoretic reason for the eigenbasis: the graph is a Cayley graph of $(\mathbb{Z}/2)^d$, and its translation-invariant Laplacian is diagonal in the characters of that group.

Example 3.6 (Cycles and periodic grids)**.** The cycle C_q, $q \geq 3$, is a Cayley graph of $\mathbb{Z}/q$. Its Fourier characters have Laplacian eigenvalues $2 - 2\cos(2\pi j/q)$. With $\zeta = e^{2\pi i/q}$,

$$\prod_{j=1}^{q-1}(2 - \zeta^j - \zeta^{-j}) = \left|\prod_{j=1}^{q-1}(1 - \zeta^j)\right|^2 = q^2.$$

The last equality follows by evaluating $(x^q - 1)/(x - 1)$ at $x = 1$. The spectral formula therefore gives $\tau(C_q) = q$. With arbitrary conductances, a tree is obtained by omitting exactly one cycle edge, so the refined identity is

$$\tau(C_q; c) = \left(\prod_e c_e\right)\sum_e \frac{1}{c_e}.$$

The same Fourier calculation extends to a two-dimensional periodic grid. For $p, q \geq 3$, the Cartesian product $C_p \,\square\, C_q$ has pq vertices, and its Laplacian is the sum of the two factor Laplacians. Product characters give

$$\tau(C_p \,\square\, C_q) = \frac{1}{pq}\prod_{\substack{0\leq a<p,\ 0\leq b<q\\(a,b)\neq(0,0)}}\left(4 - 2\cos\frac{2\pi a}{p} - 2\cos\frac{2\pi b}{q}\right).$$

For $p = q = 3$, the nonzero eigenvalues are 3 with multiplicity four and 6 with multiplicity four. Its tree count is $3^4 6^4/9 = 11\,664$. More generally, a Cartesian product of any connected graphs has eigenvalues given by all sums of factor eigenvalues: apply $L_X \otimes I + I \otimes L_Y$ to tensor products of eigenvectors. One spectral decomposition therefore supplies a family of exact enumeration formulas.

Example 3.7 (The Petersen graph)**.** Realize its ten vertices as the 2-element subsets of $\{1, 2, 3, 4, 5\}$, with adjacency meaning disjointness. Every vertex has three neighbors. Adjacent vertices have no common neighbor; distinct nonadjacent vertices have one. If P is its adjacency matrix, counting length-two walks gives

$$P^2 = 2I - P + J.$$

Constants have eigenvalue 3. On their orthogonal complement, the remaining eigenvalues solve $x^2 + x - 2 = 0$, hence are 1 and -2. Their multiplicities a, b satisfy $a + b = 9$ and $3 + a - 2b = 0$, because $\operatorname{tr} P = 0$. Thus $a = 5, b = 4$. The Laplacian $3I - P$ has spectrum

$$0, \quad 2 \text{ with multiplicity } 5, \quad 5 \text{ with multiplicity } 4,$$

and

$$\tau(\text{Petersen}) = \frac{2^5 5^4}{10} = 2000.$$

The count is an explicit consequence of the graph's uniform intersection patterns.

3.4 The inverse measures resistance

For distinct vertices u, v, put $b = \mathbf{e}_u - \mathbf{e}_v$. Injecting one unit of current at u and removing it at v means solving

$$L\phi = b.$$

The mean-zero solution is $\phi = L^+ b$. The voltage drop is the effective resistance

$$R_{uv} = b^{\mathsf{T}} L^+ b. \tag{14}$$

This definition agrees with minimizing dissipated energy. Indeed, the current $q = WB^{\mathsf{T}}\phi$ satisfies $Bq = b$, and any other such current has the form $q + h$ with $Bh = 0$. Expanding its energy gives

$$(q + h)^{\mathsf{T}} W^{-1}(q + h) = q^{\mathsf{T}} W^{-1} q + h^{\mathsf{T}} W^{-1} h,$$

because the cross term is $2\phi^{\mathsf{T}} Bh = 0$. The minimum is $\phi^{\mathsf{T}} L\phi = R_{uv}$. These electrical and tree interpretations are developed systematically in [5].

Proposition 3.8 (Adding one edge)**.** *Add a new edge from u to v of conductance $s > 0$, allowing a parallel edge if one already exists. Then*

$$\tau(G + e; c, s) = \tau(G; c)(1 + sR_{uv}), \qquad \Pr(e \in T) = \frac{sR_{uv}}{1 + sR_{uv}} \tag{15}$$

under the weighted tree distribution of the enlarged graph.

Proof. Ground a root and remove its coordinate from b, obtaining $\widehat{b}$. The new reduced Laplacian is $\widehat{L} + s\widehat{b}\widehat{b}^{\mathsf{T}}$. The rank-one determinant identity gives

$$\det(\widehat{L} + s\widehat{b}\widehat{b}^{\mathsf{T}}) = \det \widehat{L}\big(1 + s\widehat{b}^{\mathsf{T}} \widehat{L}^{-1} \widehat{b}\big).$$

Grounded and mean-zero solutions differ only by a constant, so the quadratic form is R_{uv}. The term independent of s counts trees avoiding the new edge; the coefficient of s, multiplied by s, counts those containing it. Divide by the total weight. □

For a tree, the unique unit flow follows the path from u to v, so R_{uv} is the sum of $1/c_e$ along that path. With unit conductances and path length ℓ, adding a unit edge creates $1 + \ell$ spanning trees. For a unit square, opposite vertices are joined by two length-two branches in parallel, so $R = 1$. Adding a diagonal therefore changes its tree count from 4 to 8; the new diagonal appears with probability $1/2$. These are instances of one formula, which also applies when the existing network has many intertwined cycles.

Increasing a conductance can only decrease every effective resistance. This follows from the energy minimum, or directly by differentiating the inverse on $\mathbf{1}^{\perp}$:

$$\frac{\partial R_{uv}}{\partial c_e} = -\left(b_e^{\mathsf{T}} L^{+} b\right)^2 \le 0.$$

The right side also measures the sensitivity. If the endpoints of e have equal potential under this unit-current experiment, increasing that conductance has zero first-order effect. For an edge added directly between u and v, the rank-one inverse identity gives the exact new resistance $R_{uv}/(1+sR_{uv})$. Enumeration and electrical response are controlled by the same matrix perturbation.

3.5 The Hodge projection becomes a probability kernel

Choose a random spanning tree with

$$\Pr(T) = \frac{\prod_{e\in T} c_e}{\tau(G;c)}.$$

For an edge e, let b_e denote its incidence column. The matrix from (11) has entries

$$K_{ef} = \sqrt{c_e c_f}\, b_e^{\mathsf{T}} L^{+} b_f.$$

Changing an edge orientation changes its row and column signs together; principal determinants are unaffected.

Theorem 3.9 (Transfer-current determinantal law)**.** *For every set S of distinct edges,*

$$\Pr(S \subseteq T) = \det K_S. \tag{16}$$

In particular, $\Pr(e \in T) = c_e R_{s(e)t(e)}$.

Proof. Again write $Y = \widehat{B}W^{1/2}$, so $M = YY^{\mathsf{T}} = \widehat{L}$ is invertible. For $e \in S$ introduce an independent variable z_e, and multiply that edge's conductance by $1 + z_e$. The matrix-tree theorem gives the polynomial identity

$$\frac{\det(M + Y_S \operatorname{diag}(z_e) Y_S^{\mathsf{T}})}{\det M} = \sum_{T} \Pr(T) \prod_{e\in T\cap S} (1 + z_e).$$

The coefficient of $\prod_{e\in S} z_e$ on the right is precisely $\Pr(S \subseteq T)$. On the left, factor out M and use $\det(I + UV) = \det(I + VU)$ to obtain

$$\det\Big(I_S + \operatorname{diag}(z_e) Y_S^{\mathsf{T}} M^{-1} Y_S\Big).$$

Expanding by multilinearity in its rows shows that the coefficient of the full product is $\det(Y_S^{\mathsf{T}} M^{-1} Y_S)$. Grounded and mean-zero potentials differ only by a constant, which the incidence transpose kills. Hence

$$Y^{\mathsf{T}} M^{-1} Y = A^{\mathsf{T}} L^{+} A = K.$$

This proves the joint-inclusion formula; its one-edge case is $K_{ee} = c_e R_{s(e)t(e)}$. If $|S| > N - 1$, the minor has deficient rank and both sides vanish. The empty set gives 1. □

The determinantal law is associated with Burton and Pemantle [6]. Each random tree selects a basis of the incidence columns, and the projection minors give the probabilities of joint selection.

Since K is a projection of rank $N-1$,

$$\sum_e c_e R_{s(e)t(e)} = \operatorname{tr} K = N-1. \tag{17}$$

For a unit-conductance edge-transitive graph, all edge probabilities are $(N-1)/M$. They are $2/N$ for K_N, $(p+q-1)/(pq)$ for $K_{p,q}$, $7/12$ for the ordinary cube, and $3/5$ for Petersen. For a weighted triangle,

$$\tau = ab+ac+bc, \qquad \Pr(e_a \in T) = \frac{a(b+c)}{ab+ac+bc},$$

which is exactly the diagonal entry obtained from its harmonic-line projection.

For distinct edges, the 2×2 determinant gives

$$\operatorname{Cov}(\mathbf{1}_{e\in T}, \mathbf{1}_{f\in T}) = -K_{ef}^2 \le 0. \tag{18}$$

This proves pairwise negative correlation. Hadamard's inequality also gives $\Pr(S \subseteq T) \le \prod_{e\in S} \Pr(e \in T)$. If S contains a cycle, its incidence columns are dependent and the determinant vanishes, as required for a tree.

The tree polynomial also records exact deletion effects. If $p_e = \Pr(e \in T)$, then

$$\tau(G \setminus e; c) = \tau(G; c)(1-p_e).$$

A bridge has $p_e = 1$; deleting it gives zero spanning-tree weight. For a unit cycle, each edge has $p_e = (q-1)/q$, and deleting one leaves the unique path tree. An edge with inclusion probability near one accounts for most of the tree weight, so deleting it removes most of that weight.

For any specified edge set S, there is also a compact law for its random occupancy $D_S = |T \cap S|$:

$$\mathbb{E} z^{D_S} = \det(I_S + (z-1)K_S).$$

As a principal submatrix of a projection, K_S has eigenvalues $\eta_i \in [0,1]$, so this factors as $\prod_i (1-\eta_i+\eta_i z)$. Therefore D_S has the distribution of a sum of independent Bernoulli variables with parameters η_i. These variables describe the total count; the selected edges themselves generally remain dependent.

Example 3.10 (Adjacent and opposite edges in K_4)**.** Here $L = 4I - J$ and $L^+ = \frac{1}{4}(I - \frac{1}{4}J)$. Thus $K_{ee} = 1/2$. For edges sharing a vertex, $|b_e^{\mathsf{T}} b_f| = 1$, so $|K_{ef}| = 1/4$; disjoint edges give $K_{ef} = 0$. Therefore

$$\Pr(e, f \in T) = \begin{cases} 3/16, & e, f \text{ share a vertex}, \\ 1/4, & e, f \text{ are disjoint}. \end{cases}$$

Among the 16 trees, three contain a prescribed adjacent pair and four contain a prescribed opposite pair. Conditioning on one edge lowers an adjacent edge's probability from $1/2$ to $3/8$, while an opposite edge remains at $1/2$.

There is a further small surprise. Orient the three edges at a chosen vertex consistently. Their principal kernel is $\frac{1}{4}(I_3 + J_3)$, with eigenvalues $1, 1/4, 1/4$. If D is that vertex's degree in T, the same generating identity gives

$$\mathbb{E} z^D = \det(I_3 + (z-1)K_S) = z\left(\frac{3+z}{4}\right)^2.$$

Hence $\Pr(D=1) = 9/16$, $\Pr(D=2) = 6/16$, and $\Pr(D=3) = 1/16$. The full degree distribution comes from a three-dimensional projection calculation.

3.6 A sharp inequality for every marked set of edges

Mark any set of edges, and sort the trees by how many marked edges they use. The Hodge projection forces the generating polynomial's roots to be real, giving a sharp inequality for these counts even when the graph has no symmetry.

Theorem 3.11 (Sharp marked-edge inequalities)**.** *Let $G = (V, E)$ be a finite connected loopless graph, possibly with parallel edges, with positive edge weights c_e. Fix $S \subseteq E$, and let a_k be the sum of $\prod_{e\in T} c_e$ over spanning trees with $|T \cap S| = k$. Write $c(V, F)$ for the number of components of (V, F), including isolated vertices, and put*

$$\ell = c(V, E \setminus S) - 1, \qquad d = |V| - c(V, S), \qquad h = d - \ell.$$

Then $a_k > 0$ exactly for $\ell \le k \le d$. The polynomial $z^{-\ell}\sum_k a_k z^k$ has degree h and only strictly negative real roots. For every $\ell < k < d$,

$$a_k^2 \ge \frac{(k-\ell+1)(d-k+1)}{(k-\ell)(d-k)}\, a_{k-1}a_{k+1}. \tag{19}$$

Equivalently, the sequence $a_{\ell+j}/\binom{h}{j}$, $0 \le j \le h$, is log-concave. The constants are sharp, even for unweighted simple graphs.

Proof. Retain $A = BW^{1/2}$, $L = AA^{\mathsf{T}}$, and the Hodge projection $K = A^{\mathsf{T}}L^{+}A$. Giving marked edges the extra weight z, the matrix-tree theorem and the determinant identity already used for the transfer-current law give

$$F(z) := \sum_k a_k z^k = \tau(G; c)\det(I + (z-1)K_S) = \tau(G; c)\prod_i (1 - \eta_i + \eta_i z), \qquad 0 \le \eta_i \le 1.$$

For completeness, delete one vertex row from A. The changed reduced Laplacian is $\widehat{L} + (z-1)\widehat{A}_S\widehat{A}_S^{\mathsf{T}}$; factoring $\widehat{L}$ and using $\det(I+UV) = \det(I+VU)$ gives the first determinant. The rooted expression for its kernel is the same orthogonal projection K.

Since L^{+} is positive definite on $\mathbf{1}^{\perp}$,

$$\operatorname{rank} K_S = \operatorname{rank} A_S = |V| - c(V, S) = d.$$

An eigenvector of K_S with eigenvalue 1, extended by zero outside S, lies in $\operatorname{im} A^{\mathsf{T}}$: equality in the contraction inequality for an orthogonal projection forces the extended vector into its image. Such gradients are exactly those coming from vertex functions constant on every component of $(V, E \setminus S)$. Modulo global constants their dimension is ℓ. Conversely every such gradient gives an eigenvector with eigenvalue 1. Thus the product has ℓ factors z, h factors with $0 < \eta_i < 1$, and constant factors from the zero eigenvalues. This proves the root and support assertions.

We now derive the coefficient inequality. Write $Q(z) = z^{-\ell}F(z) = \sum_{j=0}^{h} b_j z^j$. Fix $1 \le j < h$ and set $f = Q^{(j-1)}$, of degree $n = h - j + 1$. Rolle's theorem, with root multiplicities retained, shows that all roots of f are negative. Hence $f(z) = C\prod_{i=1}^{n}(1 + t_i z)$ with $C, t_i > 0$. If its first three coefficients are C, B, D, Cauchy–Schwarz gives $B^2 \ge 2nCD/(n-1)$, because $(\sum_i t_i)^2 \le n\sum_i t_i^2$ and $2\sum_{i<q} t_i t_q = (\sum_i t_i)^2 - \sum_i t_i^2$. Substitute $C = (j-1)!b_{j-1}$, $B = j!b_j$, and $D = (j+1)!b_{j+1}/2$ to obtain

$$b_j^2 \ge (j+1)(h-j+1)b_{j-1}b_{j+1}/(j(h-j)),$$

which is (19). Sharpness follows from the next example. □

Example 3.12 (An infinite equality family and a strict case)**.** For K_N, $N \geq 2$, mark all edges incident to one vertex and orient them outward. Then $K_S = (I + J)/N$, where J is the $(N-1) \times (N-1)$ all-ones matrix. Its eigenvalues are 1 and $1/N$, the latter repeated $N-2$ times. Consequently

$$F(z) = z(N - 1 + z)^{N-2}, \qquad a_{1+j} = \binom{N-2}{j}(N-1)^{N-2-j}.$$

The normalized coefficients form a geometric progression, so every interior inequality is an equality. For K_4, the counts are $9, 6, 1$: the sharp factor is 4, and $6^2 = 4 \cdot 9 \cdot 1$. Instead mark the three edges of a Hamilton path in K_4. The marked and unmarked edges each form a path, and their kernel has eigenvalues $1/2, (2-\sqrt{2})/4, (2+\sqrt{2})/4$. Thus $F(z) = 1 + 7z + 7z^2 + z^3$, and the sharp inequality gives $49 \geq 3 \cdot 1 \cdot 7$ at both interior indices.

This is Newton's inequality realized by a graph's Hodge projection. It applies to every marked edge set and every choice of positive weights. The chromatic-polynomial theorem later requires a richer intersection ring and its deep Hodge–Riemann theorem; the argument here closes using the operators we have already constructed.

3.7 A determinant sensitive to holonomy

A twisted version replaces scalar differences by parallel transport. Assign $\rho_e \in U(1)$ to each oriented edge, with $\rho_{\bar{e}} = \rho_e^{-1}$, and define

$$(d_\rho f)(e) = f(t(e)) - \rho_e f(s(e)), \qquad L_\rho = d_\rho^* W d_\rho.$$

Vertex changes of unitary basis conjugate this Laplacian and preserve the holonomy $w_\gamma = \prod_{e \in \gamma} \rho_e^{\pm 1}$ around every oriented cycle. Forman's theorem, reproved by Kenyon [7], states

$$\det L_\rho = \sum_{F \text{ cycle-rooted spanning forest}} \left(\prod_{e \in F} c_e \right) \prod_{\gamma \text{ cycle of } F} |1 - w_\gamma|^2. \tag{20}$$

Each component contains exactly one cycle. Its weight combines edge conductances with cycle holonomy. The next calculation proves the formula directly for a single cycle.

For a cycle with $N \geq 3$ vertices, orient the edges coherently and put $w = \rho_1 \cdots \rho_N$. The square matrix d_ρ has only two nonzero terms in its determinant expansion: choosing all forward vertex entries, or choosing all source entries. Their signed difference gives

$$|\det d_\rho|^2 = |1 - w|^2, \qquad \det L_\rho = \left(\prod_i c_i \right) |1 - w|^2.$$

With unit conductances and $w = e^{i\theta}$, a gauge makes every edge transport $e^{i\theta/N}$. The ordinary Fourier modes then give eigenvalues

$$2 - 2\cos\left(\frac{2\pi j - \theta}{N}\right), \qquad j = 0, \ldots, N-1,$$

whose product is $2 - 2\cos\theta$. Trivial holonomy restores the constant section and the zero eigenvalue. Nontrivial holonomy removes that section. This is a concrete instance in which keeping monodromy turns a degeneracy into a measurable determinant.

The incidence map has given us counts, resistances, and correlations. Keeping its integral lattices will reveal a further invariant: the finite graph Jacobian.

4 The integral structure: trees, lattices, and a finite Jacobian

Working over a field reveals the number of independent cycles. Working over $\mathbb{Z}$ also reveals lattice indices: do the boundary columns generate the whole lattice of possible boundaries, or a sublattice of finite index? For graphs the index is one. In higher dimensions it can carry torsion.

Let $G = (V, E)$ be a finite connected undirected graph, with an orientation chosen for each edge, and let $B : \mathbb{Z}^E \to \mathbb{Z}^V$ be its incidence map. A loop has zero column. Write

$$\mathbb{Z}_0^V = \left\{ (a_v)_{v \in V} : \sum_{v \in V} a_v = 0 \right\}.$$

If r is a fixed vertex, a path from r to v, with signs adjusted to the chosen orientations, has boundary $v - r$. These differences form an integral basis of $\mathbb{Z}_0^V$. Thus connectedness gives integral surjectivity, and hence surjectivity after tensoring with $\mathbb{Q}$:

$$0 \longrightarrow H_1(G; \mathbb{Z}) \longrightarrow \mathbb{Z}^E \xrightarrow{B} \mathbb{Z}^V \xrightarrow{\sum} \mathbb{Z} \longrightarrow 0. \tag{21}$$

There are no two-cells, so the kernel really is H_1. Consequently it is free of rank $|E| - |V| + 1$. If $|E| = |V| - 1$, the map $B : \mathbb{Z}^E \to \mathbb{Z}_0^V$ is an isomorphism. Deleting the row of r therefore gives a square matrix of determinant ± 1. A tree solves every integral conservation equation uniquely, with integral edge values.

Lemma 4.1 (The incidence matrix is totally unimodular)**.** *Every square minor of B belongs to $\{0, 1, -1\}$.*

Proof. Consider a square submatrix. If a column has at most one nonzero entry, expansion along that column reduces the assertion to a smaller minor. Otherwise every column has its two entries $+1, -1$, and the sum of the rows is zero, so the determinant vanishes. Starting with the empty minor of determinant one proves the assertion. □

This elementary argument explains why graph boundary bases have no hidden integral index. It also explains a familiar computational advantage: incidence constraints often preserve integrality in optimization problems. Higher cellular boundary matrices need not be totally unimodular.

4.1 A group whose order counts spanning trees

Put $L = BB^{\mathsf{T}}$, and delete the row and column of r to obtain $L_r = B_r B_r^{\mathsf{T}}$. The finite graph Jacobian, also called the critical or sandpile group, is

$$\operatorname{Jac}(G) = \mathbb{Z}_0^V / L\mathbb{Z}^V \cong \mathbb{Z}^{V \setminus \{r\}} / L_r \mathbb{Z}^{V \setminus \{r\}}.$$

Indeed, the sum of the columns of L is zero, so its non-root columns generate its image. Projection off the root identifies $\mathbb{Z}_0^V$ with $\mathbb{Z}^{V \setminus \{r\}}$ and sends these generators to the columns of L_r. This also makes the independence of the root transparent. The terminology connects linear equivalence of vertex divisors with chip-firing: firing a vertex changes a divisor by a Laplacian column [8].

Proposition 4.2 (Integral refinement of the matrix-tree theorem)**.** *For a connected graph,*

$$|\operatorname{Jac}(G)| = \det L_r = \tau(G),$$

where $\tau(G)$ is its number of spanning trees. For the one-vertex graph the reduced matrix is empty and its determinant is one.

Proof. Loops have zero incidence columns and belong to no spanning tree, so removing them preserves both L_r and $\tau(G)$. The quadratic form $x^{\mathsf{T}} L_r x = \|B_r^{\mathsf{T}} x\|^2$ is positive definite: extend x by zero at the root, and connectedness forces a function with zero edge differences to vanish. The integer matrix L_r has nonzero determinant. Integer row and column operations with determinant ± 1 put it in Smith normal form $\operatorname{diag}(d_1, \dots, d_{n-1})$. They change neither the isomorphism class nor the order of its cokernel. The latter order is $\prod_i |d_i| = |\det L_r|$. The determinant is positive by the quadratic form, and Theorem 3.2 identifies it with $\tau(G)$. For one vertex the group is trivial and the empty determinant is one. The perfect-complex account in Section 2.3 organizes the same index calculation for more general complexes. □

The determinant remembers only the order. Smith normal form remembers the group: if unimodular row and column operations give $\operatorname{diag}(d_1, \dots, d_{n-1})$, with positive $d_i \mid d_{i+1}$, then

$$\operatorname{Jac}(G) \cong \bigoplus_i \mathbb{Z}/d_i\mathbb{Z}.$$

Example 4.3 (Cycles and complete graphs)**.** For a cycle C_m, $m \geq 3$, $\operatorname{Jac}(C_m) \cong \mathbb{Z}/m\mathbb{Z}$. To see the group directly, number the vertices cyclically and write $x_i = [v_i - v_0]$, with $x_0 = x_m = 0$. The Laplacian relations say $2x_i = x_{i-1} + x_{i+1}$, hence $x_i = ix_1$ and $mx_1 = 0$. Conversely these substitutions satisfy every relation, including the relation at the root. The reduced Smith form is therefore $\operatorname{diag}(1, \dots, 1, m)$.

For K_n, $n \geq 2$, the reduced matrix is $nI_{n-1} - J_{n-1}$, where J is the all-ones matrix. Its presentation has generators $e_1, \dots, e_k$, $k = n-1$, and relations $ne_i - s = 0$, where $s = \sum e_j$. Change the integral basis to $d_i = e_i - e_k$, $i < k$, and e_k. Subtracting the last relation from the others gives $nd_i = 0$, while the last relation becomes $e_k - \sum_{i<k} d_i = 0$. Eliminate e_k to obtain

$$\operatorname{Jac}(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}, \qquad \operatorname{SNF}(L_r) = \operatorname{diag}(1, n, \dots, n).$$

Thus K_4 has sixteen spanning trees and Jacobian $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It has the same tree count as C_{16}, whose Jacobian is cyclic of order sixteen. Even the finite group sees more than the determinant. The case K_1 has trivial Jacobian.

4.2 The same group between cycles and cuts

Equip $\mathbb{Z}^E$ with the standard edge inner product and define

$$F = \ker B, \qquad C = \operatorname{im} B^{\mathsf{T}}.$$

These are the integral cycle and cut lattices. Over $\mathbb{R}$ they are orthogonal complements. Each is primitive in $\mathbb{Z}^E$: this is immediate for a kernel, while an integral real gradient has an integral potential after its root value is normalized to zero, by integration along paths. For an integral lattice Λ, let

$$\Lambda^{\#} = \{x \in \Lambda \otimes \mathbb{R} : \langle x, \Lambda \rangle \subseteq \mathbb{Z}\}.$$

Proposition 4.4 (A finite integral gluing obstruction)**.** *There are isomorphisms of finite abelian groups*

$$\operatorname{Jac}(G) \cong C^{\#}/C \cong \mathbb{Z}^E/(C + F) \cong F^{\#}/F.$$

In particular the determinant of either lattice is $\tau(G)$.

Proof. The vectors $B^{\mathsf{T}}v$, for non-root vertex basis vectors v, form a basis of C with Gram matrix L_r; its discriminant group is the cokernel of that matrix. Orthogonal projection $p_C : \mathbb{Z}^E \to C \otimes \mathbb{R}$ has image $C^{\#}$. For the nontrivial inclusion, a homomorphism $C \to \mathbb{Z}$ extends to $\mathbb{Z}^E \to \mathbb{Z}$, since C is primitive and hence a direct summand as an abelian group. An integral vector represents the extended functional, and its projection represents the original one. The kernel of p_C is F, and the preimage of C is $C + F$. Interchanging C and F completes the proof. The order of a discriminant group is its Gram determinant. □

This is the lattice description developed by Bacher, de la Harpe, and Nagnibeda [9]. The real orthogonal decomposition always splits, but its integral version can fail to split; the Jacobian measures that failure. For C_m, a cycle generator has squared norm m, so its rank-one discriminant group is visibly $\mathbb{Z}/m\mathbb{Z}$. For a tree, $F = 0$ and $C = \mathbb{Z}^E$, so the obstruction disappears.

5 Higher trees: the same exactness, with new arithmetic

Let X be a finite cell complex of dimension $d \geq 1$. Use the augmented integral cellular chain complex, with $\partial_0 : C_0(X;\mathbb{Z}) \to \mathbb{Z}$ the sum of coefficients. Thus degree zero homology below means *reduced* homology. Suppose $\widetilde{H}_{d-1}(X;\mathbb{Q}) = 0$. A spanning d-tree is a subcomplex T containing the entire $(d-1)$-skeleton and satisfying

$$H_d(T;\mathbb{Q}) = 0, \qquad \widetilde{H}_{d-1}(T;\mathbb{Q}) = 0.$$

For $d = 1$ these are ordinary spanning trees. For $d > 1$, the definition constrains the top two homology groups. When all lower reduced rational homology of X also vanishes, the trees are rationally acyclic.

Put $Z_{d-1} = \ker \partial_{d-1}$, a free integral lattice of rank r. A tree has exactly r top cells: its boundary columns are a basis over $\mathbb{Q}$ of $Z_{d-1} \otimes \mathbb{Q}$. Equivalently, any two of the following conditions imply the third: top-degree acyclicity, codimension-one rational vanishing, and the count r. This is the exact linear-algebraic continuation of the connected graph theorem. The boundary columns also define a representable matroid; its bases are the spanning trees under the stated vanishing assumption [10].

5.1 Why torsion is squared in the count

Choose an integral basis of the cycle lattice Z_{d-1} and express the top boundary in it as $A : \mathbb{Z}^{f_d} \to \mathbb{Z}^r$. A nonsingular r-column submatrix A_T is exactly a spanning tree. Its cokernel is

$$Z_{d-1}/\partial_d C_d(T;\mathbb{Z}) = \widetilde{H}_{d-1}(T;\mathbb{Z}) =: F_T.$$

Smith normal form therefore gives $|\det A_T| = |F_T|$. This is where the new integer information enters. A graph tree has index one; a higher tree need not.

Now put $W = \operatorname{diag}(w_\sigma)$ with positive cell weights. The same Cauchy–Binet calculation used for graph trees gives

$$\det(AWA^{\mathsf{T}}) = \sum_{|S|=r} (\det A_S)^2 \prod_{\sigma \in S} w_\sigma = \sum_T |\widetilde{H}_{d-1}(T;\mathbb{Z})|^2 \prod_{\sigma \in T_d} w_\sigma. \tag{22}$$

The square comes from multiplying a boundary minor by the same minor of its transpose. Its arithmetic meaning comes from the cokernel just computed. This proves the weighted identity without assuming a theory of determinant lines.

There is a useful second interpretation. The two-term free complex $[C_d(T;\mathbb{Z}) \to Z_{d-1}]$, placed in degrees $-1, 0$, resolves F_T. Its shifted derived dual resolves $F_T^\vee$; the universal coefficient theorem identifies the latter with $H^d(T;\mathbb{Z})$. Thus

$$|\widetilde{H}_{d-1}(T;\mathbb{Z})|\,|H^d(T;\mathbb{Z})| = |F_T|^2.$$

Section 2.3 explains how this paired arithmetic index behaves in exact triangles. It is additional structure on the already established determinant identity.

The integral cycle basis fixes the volume normalization. Using the cellular metric on C_{d-1} instead introduces the covolume and torsion corrections of the general matrix-tree theorem [11, 32]. The broader torsion–tree theorem expresses Reidemeister torsion through alternating products of such enumerations [32].

5.2 Cayley's formula in every dimension

Let $K_{n,d}$ be the complete d-skeleton on n labelled vertices, where $1 \le d \le n-1$. Its spanning trees retain every face of dimension less than d, and have $\binom{n-1}{d}$ d-faces. Kalai's theorem is

$$\boxed{\sum_{\substack{T \subseteq K_{n,d} \\ T \text{ a spanning } d\text{-tree}}} |\widetilde{H}_{d-1}(T;\mathbb{Z})|^2 = n^{\binom{n-2}{d}}.} \tag{23}$$

The binomial coefficient is zero if $d = n-1$. The formula includes Cayley's count for $d = 1$, but in higher dimensions it is a torsion-weighted count [12].

The complete simplex gives a split Koszul resolution. Its contracting homotopy will determine the determinant-line volume and the power of n. Distinguish one root vertex. The cycles $z_I = \partial[\text{root}, I]$, indexed by d-element subsets I of the other vertices, form an integral basis of Z_{d-1}. Independence follows from the unique non-root face $[I]$ in each cycle; generation follows from the cone contraction $c = \partial(\text{root} * c)$ for a cycle c. The top boundary matrix in that basis is $A = [I \mid D]$: root-containing top faces give the identity, and D is the simplex boundary matrix on the remaining $n-1$ vertices.

Identify its chain groups with exterior powers of $\mathbb{R}^{n-1}$. Writing $u = (1, \dots, 1)$, the boundary operator is contraction ι_u, and its adjoint is exterior multiplication ε_u. The Koszul relation

$$\iota_u\varepsilon_u + \varepsilon_u\iota_u = \|u\|^2 I$$

gives a contracting homotopy $\varepsilon_u/\|u\|^2$ for the full augmented simplex complex. It also controls the metric on each exterior degree. Thus

$$AA^{\mathsf{T}} = I + \iota_u\varepsilon_u \quad \text{on } \bigwedge^d \mathbb{R}^{n-1}.$$

Decompose $\mathbb{R}^{n-1} = \mathbb{R}u \oplus u^\perp$. On $\bigwedge^d u^\perp$, the operator $\iota_u\varepsilon_u$ is multiplication by $\|u\|^2 = n-1$; on $u \wedge \bigwedge^{d-1} u^\perp$, it is zero. Hence the eigenvalues of AA^{T} are n, with multiplicity $\binom{n-2}{d}$, and 1, with multiplicity $\binom{n-2}{d-1}$. Taking the determinant in (22) proves (23). The full simplex's permutation symmetry and its exterior algebra have organized an entire family of tree enumerations.

Example 5.1 (Four higher trees on four vertices)**.** For $n = 4, d = 2$, there are four triangular faces. Every tree has three of them: remove any one face from the tetrahedron's boundary. The result is a triangulated disk, so its homology has no torsion. Formula (23) gives $4^{\binom{2}{2}} = 4$, the ordinary count in this small case.

Example 5.2 (A higher tree that remembers the number two)**.** On vertices $1, \dots, 6$, take all fifteen edges and the ten faces

$$123,\ 124,\ 135,\ 146,\ 156,\ 236,\ 245,\ 256,\ 345,\ 346.$$

Each edge belongs to two faces and every vertex link is a five-cycle. Thus this is a connected closed triangulated surface with Euler characteristic $6 - 15 + 10 = 1$, namely $\mathbb{R}P^2$.

The torsion is visible without appealing to the surface classification. Use the integral cycle basis $z_{ij} = \partial[1, i, j]$, $2 \le i < j \le 6$, for the complete one-skeleton. The first five faces kill $z_{23}, z_{24}, z_{35}, z_{46}, z_{56}$. The other five impose

$$z_{36} = z_{26}, \quad z_{45} = z_{25}, \quad z_{25} = z_{26}, \quad z_{34} = -z_{45}, \quad z_{34} = z_{36}.$$

All remaining generators reduce to one a with $2a = 0$, and mapping a to $1 \in \mathbb{Z}/2\mathbb{Z}$ respects every relation. Therefore

$$H_1(T; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{SNF}(A_T) = \mathrm{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 2).$$

The square boundary map has full rank, so $H_2(T; \mathbb{Z}) = 0$. This is a spanning two-tree of $K_{6,2}$, but it contributes four to the weighted count, not one. The complete count is $6^6 = 46656$; it is a count with arithmetic multiplicity.

The distinction can be made completely numerical. Exhaustive exact enumeration of the $\binom{20}{10} = 184756$ possible ten-face sets gives 46608 integral-acyclic trees and twelve trees with first homology $\mathbb{Z}/2\mathbb{Z}$. The latter are all relabellings of the projective-plane triangulation above. Their automorphism group is A_5, of order 60, so orbit–stabilizer explains their number as $6!/60 = 12$. Hence the ordinary number of rational two-trees is 46620, whereas the determinant counts

$$46608 + 12 \cdot 2^2 = 46656 = 6^6.$$

These counts can be reproduced by forming the twenty boundary columns in the ten cone-cycle coordinates and computing every ten-column minor with exact integer elimination.

5.3 Acyclicity is not contractibility

The projective-plane example already distinguishes rational acyclicity from contractibility. Even *integral* acyclicity can miss a nontrivial fundamental group. Form the two-dimensional presentation complex of

$$P = \langle s, t \mid (st)^2 = s^3 = t^5 \rangle.$$

It has one vertex, two one-cells, and two two-cells. The exponent sums of the two attaching words $(st)^2 s^{-3}$ and $s^3 t^{-5}$ give

$$\partial_2 = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}, \qquad \det \partial_2 = -1.$$

Its positive-degree integral homology vanishes. Nevertheless P is the binary icosahedral group of order 120 [13]; the complex is not contractible.

For a graph, the fundamental group is free, and its abelianization detects its rank. Two-cells introduce relations, so a nontrivial perfect group can survive with trivial abelianization. Higher trees extend the boundary-map argument while exposing this distinction between homology and homotopy.

6 A probability law that survives in every dimension

The higher-tree determinant also gives a probability law. Its weights are the squared orders of finite integral homology groups. Lyons [25, 26] developed this probabilistic theory; the finite argument below allows arbitrary positive cell weights.

Keep a finite nonempty CW-complex X of dimension $d \geq 1$ satisfying

$$\widetilde{H}_{d-1}(X;\mathbb{Q}) = 0.$$

Use the augmented integral chains of Section 5, so degree-zero homology is reduced. Choose an integral basis of the cycle lattice $Z_{d-1} = \ker \partial_{d-1}$, write r for its rank, and let f be the number of d-cells. In this basis the top boundary is

$$A : C_d(X;\mathbb{Z}) \longrightarrow Z_{d-1} \cong \mathbb{Z}^r, \qquad A \in \mathbb{Z}^{r\times f},$$

and the homology hypothesis gives it full row rank. By the spanning-tree criterion in Section 5, its nonsingular r-column submatrices are exactly the spanning d-trees. For each such tree,

$$\widetilde{H}_{d-1}(T;\mathbb{Z}) \cong \mathbb{Z}^r/A_T\mathbb{Z}^r, \qquad t(T) := \left|\widetilde{H}_{d-1}(T;\mathbb{Z})\right| = |\det A_T|. \tag{24}$$

The group is finite, and the trivial group has order one.

Theorem 6.1 (Torsion-weighted cellular transfer-current law)**.** *Assign a positive real weight w_σ to each d-cell. Put*

$$W = \operatorname{diag}(w_\sigma), \qquad Y = AW^{1/2}, \qquad M = AWA^t = YY^t.$$

Then the partition function is

$$\mathcal{Z}_X(w) = \sum_T t(T)^2 \prod_{\sigma\in T} w_\sigma = \det M. \tag{25}$$

Give spanning d-trees the probability distribution

$$\Pr_w(T) = \frac{t(T)^2 \prod_{\sigma\in T} w_\sigma}{\mathcal{Z}_X(w)}.$$

The matrix

$$K = Y^t M^{-1} Y = W^{1/2} A^t (AWA^t)^{-1} AW^{1/2} \tag{26}$$

is the orthogonal projection onto the row space of Y. For every set S of d-cells,

$$\Pr_w(S \subseteq T) = \det K_S, \tag{27}$$

where K_S is the principal submatrix indexed by S. In particular, for distinct cells σ, τ,

$$\Pr_w(\sigma \in T) = K_{\sigma\sigma}. \tag{28}$$

$$\operatorname{Cov}_w(\mathbf{1}_{\sigma\in T}, \mathbf{1}_{\tau\in T}) = -K_{\sigma\tau}^2 \leq 0. \tag{29}$$

The partition function, probability distribution, and kernel are unchanged by changing the integral basis of Z_{d-1}.

Proof. Cauchy–Binet and $|\det A_T| = t(T)$ give the partition function $\det(YY^t)$. Positive weights and full row rank make $M = YY^t$ positive definite. The matrix $K = Y^t M^{-1} Y$ is symmetric, has image the row space of Y, and satisfies $K^2 = Y^t M^{-1}(YY^t)M^{-1}Y = K$. It is therefore its orthogonal projection.

To obtain the probabilities, multiply each weight indexed by S by an independent factor $1 + z_\sigma$. The determinant ratio is

$$\frac{\det(M + Y_S \operatorname{diag}(z_\sigma) Y_S^t)}{\det M} = \det(I_S + \operatorname{diag}(z_\sigma) K_S).$$

In the partition-function expansion, the coefficient of $\prod_{\sigma \in S} z_\sigma$ is $\Pr_w(S \subseteq T)$. In the determinant expansion, it is $\det K_S$. This is the same coefficient-comparison argument as for graph edges. The one-by-one and two-by-two minors give the marginal and covariance formulas.

A change of integral cycle basis replaces A by UA with $U \in \mathrm{GL}_r(\mathbb{Z})$. Its minors change by $\det U = \pm 1$, so their squares and the partition function are unchanged. Its real row space is unchanged, so K is unchanged as well. In rank zero the determinant is 1, the only selected set is empty, and the formulas retain these conventions. □

Changing the orientation of a top cell conjugates K by a diagonal sign matrix, so principal determinants are unchanged. The ensemble depends on the cellular decomposition: subdivision changes its ground set and trees. The integral cycle basis fixes the normalization of M and (25), as described above.

For $d = 1$, the hypothesis says that the graph is connected. Every spanning tree has trivial reduced H_0, so $t(T) = 1$. The theorem becomes the weighted graph matrix-tree and transfer-current laws. For $d \geq 2$, finite homology can be nontrivial; the same projection law naturally selects a different measure.

Corollary 6.2 (Adding a cell and measuring its influence)**.** *Attach an additional d-cell to X, keeping the lower skeleton fixed, and let its cellular boundary have cycle-coordinate vector $a \in \mathbb{Z}^r$. Give it weight $s > 0$. If $q = a^t M^{-1} a$, then*

$$\mathcal{Z}_{X \cup \sigma}(w, s) = \mathcal{Z}_X(w)(1 + sq), \qquad \Pr_{w,s}(\sigma \in T) = \frac{sq}{1 + sq}.$$

For an existing cell,

$$\frac{\partial \log \mathcal{Z}_X}{\partial \log w_\sigma} = \Pr_w(\sigma \in T).$$

Proof. The new matrix is $M + saa^t$. The matrix determinant lemma gives the first identity. Trees avoiding the new cell retain total weight $\mathcal{Z}_X(w)$, so their complement gives the probability stated. Differentiating the finite sum in (25) proves the final identity. □

In the graph case a is the difference of the two endpoint vectors and q is their effective resistance in the original weighted network. The same insertion formula measures how a new cell boundary changes the torsion-weighted family.

6.1 An explicit correlation formula in every dimension

Let $X_{n,d}$ be the complete d-dimensional simplicial complex on n labelled vertices, with $1 \leq d \leq n-1$, and give its top faces unit weights. Write D for the ordinary boundary matrix from oriented d-faces

to oriented $(d-1)$-faces. Its rows use cellular coordinates. Its row space is the same as that of A: writing the cycle-basis embedding as B gives $D = BA$ with B injective.

Corollary 6.3 (Complete-complex transfer-current formula)**.** *For a torsion-weighted random spanning d-tree of $X_{n,d}$,*

$$K = \frac{1}{n} D^t D, \qquad \Pr(\sigma \in T) = \frac{d+1}{n}.$$

For distinct d-faces σ, τ,

$$\Pr(\sigma, \tau \in T) = \begin{cases} \dfrac{d(d+2)}{n^2}, & \text{if } \sigma, \tau \text{ share a } (d-1)\text{-face,} \\ \dfrac{(d+1)^2}{n^2}, & \text{otherwise.} \end{cases}$$

The number of top faces in every such tree is $\binom{n-1}{d}$.

Proof. Identify the oriented simplicial chain spaces with exterior powers of $\mathbb{R}^n$. For $u = e_1 + \cdots + e_n$, the boundary is contraction $D = \iota_u$ and its adjoint is exterior multiplication $D^t = \varepsilon_u$, where $\varepsilon_u(v) = u \wedge v$. The identities

$$\iota_u^2 = \varepsilon_u^2 = 0, \qquad \iota_u \varepsilon_u + \varepsilon_u \iota_u = nI$$

give $(D^t D)^2 = n D^t D$. Thus $D^t D/n$ is the orthogonal projection onto the row space of D, which equals the row space of A and hence is the kernel in (26).

Each boundary column has $d+1$ entries equal to ± 1, so the diagonal entries of K are $(d+1)/n$. Two distinct top faces have one common boundary face exactly when they share d vertices. In that case the corresponding off-diagonal entry is $\pm 1/n$; in every other case it is zero. The two-cell determinant proves the formula. Finally, splitting $\mathbb{R}^n = \mathbb{R}u \oplus u^\perp$ shows that contraction maps $u \wedge \Lambda^d u^\perp$ isomorphically onto $\Lambda^d u^\perp$ and annihilates $\Lambda^{d+1} u^\perp$. Its rank is therefore $\binom{n-1}{d}$. □

Cells with no common boundary face are pairwise independent. For a larger collection, the joint law is the principal determinant (27). The familiar complete-graph edge probability $2/n$ belongs to the family $(d+1)/n$ in all dimensions.

6.2 Six vertices: torsion removes a positive correlation

For $X_{6,2}$ there are 20 triangles and every spanning two-tree has 10 triangles. The complete-complex kernel is

$$K = \frac{1}{6} D^t D.$$

Every triangle has inclusion probability $1/2$. Two triangles sharing an edge have joint probability $2/9$ and covariance $-1/36$. Triangles sharing just a vertex, or having disjoint vertex sets, have joint probability $1/4$ and covariance zero.

The projective-plane example gave 46608 integral-acyclic trees and twelve trees with first homology $\mathbb{Z}/2\mathbb{Z}$. The ordinary and torsion-weighted counts are therefore

$$\#\{\text{rational two-trees}\} = 46620, \qquad \sum_T t(T)^2 = 46608 + 4 \cdot 12 = 46656 = 6^6.$$

We now compare the corresponding probability measures. Under the uniform measure each of the 46620 trees has equal weight. Under the determinant measure each projective-plane triangulation has weight four. The probability of obtaining one of those twelve triangulations is therefore

$$\Pr_{\text{unif}}(\mathbb{R}P^2) = \frac{1}{3885}, \qquad \Pr_{\text{torsion}}(\mathbb{R}P^2) = \frac{1}{972}.$$

The effect reaches beyond the probability of torsion itself. Exact inclusion counts give the following comparison; both measures have one-triangle marginal $1/2$.

Triangle relation	Uniform covariance	Torsion covariance
Share an edge	$-431/15540$	$-1/36$
Share one vertex	$-1/15540$	0
Disjoint vertices	$1/5180$	0

In particular, the complementary triangles 123 and 456 occur together in 11664 uniformly counted trees. Thus

$$\Pr_{\text{unif}}(123, 456 \in T) = \frac{11664}{46620} = \frac{324}{1295}, \qquad \operatorname{Cov}_{\text{unif}}(\mathbf{1}_{123\in T}, \mathbf{1}_{456\in T}) = \frac{1}{5180} > 0.$$

Uniform rational higher trees can have positively correlated faces. Weighting by squared torsion orders gives the projection law, with zero covariance for this pair.

For reproducibility, the enumeration uses the cycle basis $c_{ij} = \partial[1ij]$, $2 \le i < j \le 6$, of the complete graph. The ten triangle columns containing vertex 1 are unit vectors; for $2 \le i < j < k \le 6$, the remaining columns are

$$\partial[ijk] = c_{jk} - c_{ik} + c_{ij}.$$

One runs through all ten-column subsets, evaluates their determinants by checked fraction-free elimination, and accumulates either weight 1 or weight $(\det A_T)^2$ for each nonsingular subset. For each fixed pair, the resulting raw and weighted counts are respectively $10362, 10368$ when the triangles share an edge; $11652, 11664$ when they share one vertex; and $11664, 11664$ when they are complementary. All pairs in a given class have the same counts. The weighted counts also follow independently from the projection formula. The computation verifies the finite counts and homology orders.

7 Cancel the coloring sum and calculate an intersection form

Coloring and tree counting ask different questions. The tree projection does not enumerate proper colorings. We first build a cancellation rule for the coloring sum, then calculate a small intersection form explicitly. These are complete finite arguments. The general Hodge theorem and its interpretation of coefficients are separate inputs, collected in Section 13; the cancellation argument does not prove general log-concavity.

7.1 A cancellation rule for the chromatic coefficients

Let G be a finite simple graph on n vertices, and order its edges. Inclusion–exclusion over monochromatic edges gives

$$\chi_G(q) = \sum_{S\subseteq E} (-1)^{|S|} q^{c(V,S)}.$$

Indeed, imposing equality at every edge of S leaves one independent color on each component. The polynomial therefore counts proper colorings for every positive integer q.

Most terms cancel in pairs. For an edge e, write $S_{<e}$ for the selected edges smaller than e. Call e available if its endpoints are connected using $S_{<e}$. If an available edge exists, choose the least one and toggle its membership in S.

Proposition 7.1 (An explicit cancellation involution)**.** *The toggle preserves $c(V,S)$, reverses the sign $(-1)^{|S|}$, and is its own inverse. Its unpaired sets are forests containing no broken circuit, where a broken circuit is a cycle with its largest edge removed. Consequently*

$$\chi_G(q) = \sum_{S \text{ contains no broken circuit}} (-1)^{|S|} q^{n-|S|}.$$

Proof. The path witnessing availability does not use e, so adding or removing e changes no component. Toggling e changes none of the tests for smaller edges, and its own test uses only smaller edges. The least available edge is therefore unchanged on the second application. Exactly one membership bit changes, so the signs are opposite.

An available edge, together with a simple witnessing path, gives a cycle whose largest edge is that edge. Its remaining path is a broken circuit contained in S. Conversely, a contained broken circuit makes the missing largest edge available. Thus no edge is available exactly when S contains no broken circuit. Such an S is a forest: the largest edge of a cycle contained in S would be available. A forest with $|S|$ edges has $n - |S|$ components, proving the formula. □

The absolute coefficient of q^{n-k} now counts specified k-edge forests, without cohomology. For a connected graph with at least one edge, let e_* be the largest edge. No broken circuit contains e_*. Pair the surviving sets according to whether they contain e_*. This gives

$$\frac{\chi_G(q)}{q(q-1)} = \sum_{\substack{S \subseteq E \setminus \{e_*\} \\ S \text{ contains no broken circuit}}} (-1)^{|S|} q^{n-2-|S|}.$$

This is also a direct description of the reduced positive coefficients. For a four-cycle with edges $0 < 1 < 2 < 3$, the only broken circuit is $\{0,1,2\}$. The surviving counts are $(1,4,6,3)$; excluding edge 3 gives $(1,3,3)$. Hence

$$\chi_{C_4}(q) = q^4 - 4q^3 + 6q^2 - 3q, \qquad \frac{\chi_{C_4}(q)}{q(q-1)} = q^2 - 3q + 3.$$

Deletion–contraction itself needs no new theory: split the colorings of $G \setminus e$ into those whose two endpoint colors differ and those whose endpoint colors agree. The second class is in bijection with the colorings of G/e, with repeated constraints discarded.

7.2 Diagonalize the four-cycle's intersection form

Here is an exact finite model of the intersection calculation for this example. Use degree-one symbols H and E_{ij} for the six unordered pairs from $\{0,1,2,3\}$. In degree two impose

$$HE_{ij} = 0, \qquad E_{ij}E_{kl} = 0 \quad (\{i,j\} \neq \{k,l\}), \qquad E_{ij}^2 = -H^2, \qquad \deg(H^2) = 1.$$

Set $x_{ij} = E_{ij}$ and $x_i = H - \sum_{j \neq i} E_{ij}$. These substitutions verify the four-cycle's flat relations directly: incomparable pairs multiply to zero, $\sum_{F \ni i} x_F = H$ for each edge i, and $\deg(x_i x_{ij}) = 1$.

The reverse calculation is equally explicit. In the flat presentation put $H = \alpha_i$ and $E_{ij} = x_{ij}$. For $k \notin \{i, j\}$, expand $HE_{ij} = \alpha_k x_{ij}$; every term is an incomparable product, so it is zero. For $j \neq i$, expansion of $Hx_i = \alpha_j x_i$ leaves only $x_i x_{ij}$. Thus $H^2 = Hx_i = x_i x_{ij}$ has degree one. Finally multiply $H = x_i + \sum_{j \neq i} E_{ij}$ by E_{ij}: it gives $E_{ij}^2 = -x_i E_{ij} = -H^2$. Distinct pair variables are incomparable. These are exactly the displayed rules, so the flag presentation reduces to seven coordinates here.

In this basis the bilinear form $Q(a, b) = \deg(ab)$ has matrix

$$\operatorname{diag}(1, -1, -1, -1, -1, -1, -1).$$

The distinguished classes become

$$\alpha = H, \qquad \beta = 3H - \sum_{i<j} E_{ij}, \qquad (\deg \alpha^2, \deg \alpha\beta, \deg \beta^2) = (1, 3, 3).$$

The same three integers were obtained independently by the forest cancellation. No general coefficient/intersection theorem was used for their agreement in this example.

The sign argument can also be checked without importing Hodge theory. Write $p = p_0 H + \sum p_{ij} E_{ij}$. If $Q(p, \beta) = 0$, then $3p_0 + \sum p_{ij} = 0$, and ordinary Cauchy–Schwarz gives

$$Q(p, p) = \frac{(\sum p_{ij})^2}{9} - \sum p_{ij}^2 \leq -\frac{1}{3} \sum p_{ij}^2 \leq 0.$$

Decomposing $\alpha = t\beta + p$ therefore proves $Q(\alpha, \alpha)Q(\beta, \beta) \leq Q(\alpha, \beta)^2$, or $3 \leq 9$. For arbitrary matroids, proving the required signature and identifying all coefficient/intersection numbers are deep theorems. Section 13 states those imported results and derives their general log-concavity consequence. They are not consequences of the cancellation involution alone.

8 Eliminate edge labels by a spanning tree

What if the values transported along an edge lie in a noncommutative group? The first task is concrete: change the vertex coordinates until as many edge labels as possible become identity elements. A spanning tree makes that operation systematic. The resulting normal form classifies the transports and their remaining changes of coordinates. Tensor reconstruction is a separate reference construction in Section 14.

Discrete integration also survives when addition is replaced by multiplication in an arbitrary group H. Assign $a_e \in H$ to an oriented edge, with $a_{\bar{e}} = a_e^{-1}$. A change of basis $h_v \in H$ at each vertex changes the labels by

$$a_e \longmapsto h_{t(e)} a_e h_{s(e)}^{-1}.$$

On a tree, choose $h_r = 1$ at a root and successively set $h_{t(e)} = h_{s(e)} a_e^{-1}$ along outward-oriented edges. Every transformed label is then 1. The unique path makes these choices consistent, even though the labels need not commute. The root normalization makes the trivialization unique.

For a general connected graph, choose a spanning tree and trivialize its labels. The remaining $r = m - n + 1$ edges carry independent holonomies. A further gauge transformation preserving the tree labels is constant on all vertices, so it conjugates these r labels simultaneously. Consequently the groupoid of edge transports and gauge transformations is equivalent to

$$[H^r / H],$$

where the brackets retain automorphisms, as well as the set of simultaneous conjugacy classes. These are precisely representations of the free group F_r in H, with changes of basis. A spanning tree exists by repeatedly removing an edge from a cycle, which preserves connectedness, until no cycle remains.

Topologically, collapsing a spanning tree turns G into a wedge of r circles. Thus $\pi_1(G) \cong F_r$, its universal cover is a tree, and $H_1(G;\mathbb{Z}) \cong F_r^{\mathrm{ab}} \cong \mathbb{Z}^r$. This supplies a second explanation of the original theorem: for a graph, vanishing first integral homology forces the free fundamental group itself to be trivial [1]. In higher dimensions, a perfect group can defeat that conclusion.

8.1 The residual label and the residual gauge

For a triangle, orient the edges $0 \to 1$, $1 \to 2$, $2 \to 0$ and label them a, b, c. Use the first two as the spanning tree. The vertex changes

$$h_0 = 1, \qquad h_1 = a^{-1}, \qquad h_2 = a^{-1}b^{-1}$$

turn the labels into $1, 1, cba$. The order cba matters: a transported vector encounters a first and c last. Changing the root frame conjugates cba. Thus trivial transport exists precisely when $cba = 1$, while isomorphism classes of transports are its conjugacy classes. An automorphism of a given normal form is an element commuting with cba.

With several chords, the same procedure leaves one label per fundamental circuit and the common centralizer as automorphism group. It is an algorithm: visit each tree edge once to determine the vertex changes, then substitute into the chord labels. Nothing requires commuting labels. The graph's topological reduction to free generators and the algebraic change of frames describe the same calculation; tensor reconstruction asks an additional question about all linear versions at once.

9 Match branches and quotient the gluing parameters

Replace vertices by connected smooth projective curves over an algebraically closed field k, and pair distinct branch points according to a connected graph Γ. The working data are now the two branches at each node and the identifications we impose between them. We calculate the matching condition and its coordinate freedom directly. The identifications with cohomology, Picard functors, motives, and nearby cycles are recorded separately in Section 14.

9.1 A node is an equality of two constant terms

The completed local ring of a node and its normalization are

$$A = k[[x, y]]/(xy), \qquad B = k[[x]] \oplus k[[y]].$$

Every element of A has a unique expression $c + xf(x) + yg(y)$. Its image in B is $(c + xf(x), c + yg(y))$. Hence there is an exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{(u,v)\mapsto v(0)-u(0)} k \longrightarrow 0.$$

Surjectivity follows from $(0, c)$; the kernel is exactly the equal-constant-term condition. For units the same calculation says $v(0)/u(0) = 1$. This local equality, repeated at the nodes, is the branch-difference or branch-ratio map in the normalization sequences. The subtraction is a tested matching rule, rather than a name for an unspecified gluing operation.

For componentwise constant data a_v, the matching matrix is the original incidence matrix: $(\delta a)_e = a_{t(e)} - a_{s(e)}$. Its kernel consists of a single common constant. Grounding a vertex and eliminating along a tree leaves $r = |E| - |V| + 1$ independent circuit discrepancies.

9.2 Line-bundle gluing is a multiplicative gauge calculation

Fix a line bundle on each normalized component. At every node choose a nonzero identification from the fiber at t_e to the fiber at s_e. After choosing reference identifications, these are scalars $\lambda_e \in k^\times$. Every automorphism of a fixed line bundle on a connected smooth projective component is a scalar, since its global invertible functions are constant. Rescaling its frame by a_v changes the edge data by

$$\lambda_e \longmapsto a_{s(e)} \lambda_e a_{t(e)}^{-1}.$$

Choose a rooted spanning tree. On an outward-oriented tree edge set $a_{t(e)} = a_{s(e)}\lambda_e$, beginning with $a_r = 1$; use the inverse rule if the edge orientation is reversed. All tree parameters become 1. The r chord parameters can be arbitrary, and no further rooted gauge transformation changes them. Without fixing the root, the remaining automorphisms are the common scalars.

Equivalently, for an integral circuit $z = (z_e)$ define

$$\gamma_z(\lambda) = \prod_e \lambda_e^{z_e}.$$

The exponent of a_v in its gauge change is zero because $\partial z = 0$. The fundamental circuits of the chosen tree show that these products classify all gluing data up to isomorphism:

$$(k^\times)^E / \delta(k^\times)^V \cong \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), k^\times).$$

The quotient is computed, including its residual automorphisms. With fixed component bundles it is a torsor for this torus; the chosen reference identifications select its origin. There is no preferred origin when those choices have not been made.

If the component bundles vary in a family, the edge parameters lie in the nonzero fibers of the corresponding Hom line bundles. The same circuit product tensors those Hom lines with powers z_e. Vertex rescalings cancel for exactly the same reason. These tensor products can vary nontrivially over the parameter space: a constant graph does not make the family of gluing torsors constant. The Picard and attachment-map theorems in the reference section identify that family globally; the gauge computation alone does not classify all such algebraic families.

9.3 An attachment becomes a three-column period test

There is a second explicit test over $\mathbb{C}$. Write an elliptic curve analytically as $\mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ and real-linearly independent periods. An attachment point is represented by $q \in \mathbb{C}$ modulo Λ. Form a rational vector space with basis e_1, e_2, e_0, the subspace $V_{\mathbb{Q}} = \mathbb{Q}e_1 + \mathbb{Q}e_2$, and the complex linear map

$$\psi : H_{\mathbb{C}} \longrightarrow \mathbb{C}, \qquad \psi = (\lambda_1 \ \lambda_2 \ q).$$

Keep the filtration $0 \subset V_{\mathbb{Q}} \subset H_{\mathbb{Q}}$ and the distinguished subspace $F^0 H_{\mathbb{C}} = \ker \psi$. A splitting compatible with these data is a rational lift of e_0 lying in F^0. Every such lift has the form $e_0 + ae_1 + be_2$, so the test is exactly

$$q + a\lambda_1 + b\lambda_2 = 0, \qquad a, b \in \mathbb{Q}.$$

It is solvable precisely when $q \in \mathbb{Q}\Lambda$, or equivalently when the point $q + \Lambda$ is torsion. Replacing the representative q by $q + m\lambda_1 + n\lambda_2$ just changes the lift by integral columns and does not change this answer.

The underlying short exact sequence of rational vector spaces always splits. The obstruction here is compatibility with the retained complex subspace, not a failure to choose a rational basis. Section 14 identifies these explicit filtered data with the Hodge realization of the nodal elliptic attachment and states the extra theorems needed to translate the calculation into motivic Galois groups. The three-column test establishes the splitting criterion without using that motivic identification as its proof.

10 The tree determinant becomes a graph hypersurface

A different construction retains how the cycle lattice sits in the edge lattice. For a finite connected graph G, choose an integral cycle basis and write its edge coefficients as the rows of an $r \times m$ matrix Z. Introduce variables α_e and restrict the diagonal quadratic form $\sum_e \alpha_e x_e^2$ to the cycle space. Its determinant is

$$\Psi_G(\alpha) = \det\bigl(Z \operatorname{diag}(\alpha_e) Z^{\mathsf{T}}\bigr).$$

An integral basis change multiplies this determinant by the square of a determinant ± 1, so the polynomial is independent of the cycle basis. It is homogeneous of degree r. This construction is developed in [18, §1–2].

Proposition 10.1 (The complementary spanning-tree formula)**.**

$$\Psi_G(\alpha) = \sum_{T \text{ spanning tree of } G} \prod_{e \notin T} \alpha_e.$$

Proof. Cauchy–Binet expands the determinant as

$$\sum_{\substack{S \subseteq E \\ |S| = r}} \det(Z_S)^2 \prod_{e \in S} \alpha_e.$$

The matrix Z_S represents projection of the cycle lattice to its coordinates on S. Its kernel is the cycle lattice of the spanning subgraph $T = E \setminus S$. Since T has $n - 1$ edges, this projection is nonsingular exactly when T is a tree. For such a tree, each edge $s \in S$ closes a unique circuit with the path in T between its endpoints. Those fundamental circuits form an integral cycle basis whose coordinates on S form the identity matrix. Thus $\det(Z_S) = \pm 1$ for a tree and 0 otherwise. Substitution gives the formula. □

This gives an exact bridge back to Section 3:

$$\Psi_G(\alpha) = \left(\prod_{e \in E} \alpha_e \right) \tau(G; \alpha^{-1}).$$

For nonzero edge parameters, substitute $c_e = \alpha_e^{-1}$ into the weighted tree polynomial. Each tree's product cancels its own edge factors, and the resulting identity extends polynomially. Thus the enumeration polynomial also defines an algebraic hypersurface, whose geometry will lead to period pairings.

This is the first Symanzik graph polynomial. Some authors use "Kirchhoff polynomial" for a spanning-tree product over edges *in* the tree. Our convention takes the product over edges *outside* it, so its degree is the cycle rank.

Example 10.2 (Three small cases)**.** For a tree, $r = 0$ and the empty determinant is 1. The sole spanning tree is G itself, whose complementary product is also 1. For a triangle,

$$\Psi_G = \alpha_1 + \alpha_2 + \alpha_3.$$

For the graph with two vertices and three parallel edges, take cycles $e_1 - e_3, e_2 - e_3$. The quadratic-form matrix is

$$\begin{pmatrix} \alpha_1 + \alpha_3 & \alpha_3 \\ \alpha_3 & \alpha_2 + \alpha_3 \end{pmatrix},$$

so

$$\Psi_G = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3.$$

Its projective zero locus is a smooth conic over $\mathbb{Q}$, with a rational point and hence isomorphic to $\mathbb{P}^1$. These equations come directly from the cycle lattice and its edge-coordinate functionals.

Because Ψ_G has integral coefficients, its affine zero locus $X_G \subset \mathbb{A}^m$ and complement U_G are algebraic varieties over $\mathbb{Q}$. They carry Betti, de Rham and ℓ-adic cohomology and motivic structures. For a tree, X_G is empty and $U_G = \mathbb{A}^m$, including $m = 0$. For a triangle,

$$U_G \cong \mathbb{A}^2 \times \mathbb{G}_m.$$

Its first cohomology is $\mathbb{Q}(-1)$, of weight 2. This differs from the weight-0 graph contribution inside a nodal curve: the two constructions attach different algebraic objects to the same graph.

10.1 Deletion–contraction before taking classes

Take an edge e that is neither a loop nor a bridge. Partitioning the spanning trees according to whether they contain e gives $\Psi_G = \alpha_e F + H$, with $F = \Psi_{G\setminus e}$ and $H = \Psi_{G/e}$. Inside $\mathbb{A}^{m-1}$ put

$$O = \{F \neq 0\} = U_{G\setminus e}, \qquad Z_0 = \{F = 0, H \neq 0\}, \qquad B = \mathbb{A}^{m-1} \setminus (V(F) \cap V(H)).$$

Thus O is open in B and Z_0 is its closed complement.

Proposition 10.3 (Eliminate one edge parameter)**.** *Projection away from α_e partitions U_G into $O \times \mathbb{G}_m$ over $F \neq 0$ and $Z_0 \times \mathbb{A}^1$ over $F = 0, H \neq 0$. It has no fiber over $F = H = 0$.*

Proof. If $F \neq 0$, set $u = \alpha_e + H/F$. The condition $\alpha_e F + H \neq 0$ is exactly $u \neq 0$; the inverse is $\alpha_e = u - H/F$. If $F = 0, H \neq 0$, the condition is true for every α_e. If both vanish, it is false for every α_e. These substitutions prove all three statements. □

Over a finite field with q elements this gives an actual counting algorithm:

$$\#U_G(\mathbf{F}_q) = (q-1)\#O(\mathbf{F}_q) + q\#Z_0(\mathbf{F}_q).$$

For each assignment to the other edges, evaluate just F and H and select the corresponding fiber size. The common-zero stratum matters: the separate counts of the deleted and contracted complements do not determine their intersection. The same partition gives the following scissor-class identity.

Taking scissor classes gives

$$[U_G] = \mathbb{L}[Z_0] + (\mathbb{L}-1)[O] = \mathbb{L}[B] - [U_{G\setminus e}], \qquad \mathbb{L} = [\mathbb{A}^1].$$

This is the deletion–contraction identity in the Grothendieck ring of varieties [19, Theorem 3.8]. The coordinate partition proves the counting and scissor identities. A motivic localization triangle retains additional connecting maps; it is stated, with its imported localization input, in Section 15. Neither the point count nor the partition alone certifies a mixed Tate motive.

11 Eliminate variables in a graph integral

The determinant gives a rational function we can now integrate. Let G be connected with cycle rank $r \geq 1$ and $m = 2r$ edges. On the affine chart $\alpha_m = 1$, consider the nonnegative integral

$$P(G) = \int_{(0,\infty)^{m-1}} \frac{d\alpha_1 \cdots d\alpha_{m-1}}{\Psi_G(\alpha_1, \ldots, \alpha_{m-1}, 1)^2}.$$

In the calculations below assume this integral is finite. The usual sufficient primitive conditions, its invariant projective form, and the relative-cohomology construction are external results recorded in Section 15. We do not infer convergence or a global period classification from the next substitutions.

11.1 The first integration is a rational elimination

Choose $e \neq m$ that is neither a loop nor a bridge. The spanning-tree partition gives $\Psi_G = tF + H$, where $t = \alpha_e$, $F = \Psi_{G\setminus e}$ and $H = \Psi_{G/e}$. For positive remaining parameters, both polynomials are positive: both graphs have spanning trees and each term in their tree sums is positive. Therefore the change $u = tF/H$ gives

$$\int_0^\infty \frac{dt}{(tF+H)^2} = \frac{1}{FH}\int_0^\infty \frac{du}{(1+u)^2} = \frac{1}{FH}.$$

Tonelli's theorem permits this integration first because the integrand is nonnegative; the assumed finiteness ensures that the resulting integral is finite. Thus one graph variable disappears and leaves a product of two explicitly computable polynomials. The deleted and contracted graphs have become the two denominators of an actual integration step.

For the graph with two vertices and two parallel edges, $r = 1$ and $\Psi = \alpha_1 + \alpha_2$. This step is the complete integral:

$$P(G) = \int_0^\infty \frac{dt}{(t+1)^2} = 1.$$

Its convergence is checked directly, without appealing to a general graph-period theorem.

11.2 The next step produces a logarithm

Each of F and H is at most linear in every remaining edge variable. For a second variable s, suppose at the fixed remaining positive parameters they have the form $F = as + b$ and $H = cs + d$, with $a, b, c, d > 0$. Partial fractions give

$$\int_0^\infty \frac{ds}{(as+b)(cs+d)} = \frac{\log(ad/bc)}{ad-bc} \qquad (ad \neq bc).$$

Indeed the antiderivative is $(ad-bc)^{-1}\log((as+b)/(cs+d))$; subtracting its endpoint values gives the formula. If $ad=bc$, the two factors are proportional and direct integration instead gives

$$\int_0^\infty \frac{ds}{(as+b)(cs+d)} = \frac{1}{ad}.$$

This is also the continuous limit of the logarithmic expression. The displayed denominator $ad-bc$ is therefore not, by itself, a singularity of the positive integral. Cases with a zero coefficient require a separate endpoint check; the formula has not asserted convergence for those cases.

The first step stayed rational; the next produced a logarithm and the polynomial $ad-bc$. Later integrations must control the resulting factors and endpoint values. This is where an edge ordering can help or fail. Brown's linear-reduction theorem supplies a general sufficient condition for particular graph families, but proving that theorem or the wheel evaluations is additional work. They belong to the reference layer, rather than being consequences of writing Ψ as a determinant.

12 What became a calculation

The incidence calculation, grounding, Cauchy–Binet expansion, inverse energy problem, and coefficient extraction give a complete finite route from edge differences to tree counts and joint probabilities. Smith operations then recover integral information, and the same minor expansion explains the squared torsion weights in higher dimensions.

The later core arguments also perform their own changes of working object. A least-edge toggle cancels the coloring sum and leaves named forests. A change of seven coordinates diagonalizes the four-cycle's intersection form. Tree elimination reduces noncommutative transport to circuit labels and their centralizers. A node becomes an equality of two constant terms; line-bundle identifications become a circuit-product quotient. Three period columns turn rational splitting into a lattice-membership test. Finally, eliminating an edge parameter gives both a fiber-counting rule and a rational integration step; eliminating the next can produce a logarithm.

These calculations have bounded conclusions. They do not prove general Hodge–Riemann positivity, identify all algebraic families with their realizations, or evaluate all graph periods. The following reference sections retain those stronger results and their essential imported hypotheses. They are further interpretations and theorems, not stages every preceding calculation had to pass through.

The shared practice is to choose operations that expose the test, perform them, and state exactly what was decided. The root-coordinate calculations in *Fourier, Symmetry, and Reconstruction* follow the same practice with different working data.

13 Reference layer: coloring spaces and Hodge positivity

The core coloring cancellation and four-cycle matrix calculation are complete without this section. Here we retain the stronger geometric realizations and general inequalities, with their imported inputs explicit: mixed Hodge theory and purity, motivic Gysin localization, and the two Adiprasito–Huh–Katz assertions about signatures and coefficient/intersection numbers. Huh proved the graph inequality using algebraic geometry [24]; Adiprasito, Huh, and Katz established the relevant Hodge theory for arbitrary matroids [23].

13.1 Deletion–contraction as a derived triangle

The chromatic coefficients can be realized as dimensions of cohomology spaces. Give each vertex a complex coordinate, and regard an edge as a forbidden equality. The resulting *coloring space* is the smooth variety

$$\mathcal{C}_G = \{(z_v) \in \mathbb{A}^V : \ z_u \neq z_v \text{ for every edge } uv\}.$$

Its defining equations have rational coefficients. A point over a finite field is a proper coloring by elements of that field. Over $\mathbb{C}$, the same equations give a hyperplane-arrangement complement. Write $P_G(t) = \sum_k \dim_{\mathbb{Q}} H^k(\mathcal{C}_G, \mathbb{Q}) t^k$. Parallel edges repeat a constraint and change neither the space nor the chromatic polynomial. A loop makes both the coloring space empty and the chromatic polynomial zero. We therefore work with the simple graph obtained by removing repeated parallel constraints.

Theorem 13.1 (Chromatic coefficients as Tate cohomology)**.** *For a finite simple graph on $n \geq 1$ vertices, if $\chi_G(q) = \sum_{k=0}^{n} (-1)^k b_k q^{n-k}$, then*

$$H^k(\mathcal{C}_G, \mathbb{Q}) \cong \mathbb{Q}(-k)^{b_k}, \qquad P_G(t) = (-t)^n \chi_G(-1/t).$$

For each edge e, there is a natural short exact sequence of mixed Hodge structures

$$0 \longrightarrow H^k(\mathcal{C}_{G\setminus e}, \mathbb{Q}) \longrightarrow H^k(\mathcal{C}_G, \mathbb{Q}) \xrightarrow{\mathrm{Res}_e} H^{k-1}(\mathcal{C}_{G/e}, \mathbb{Q})(-1) \longrightarrow 0. \tag{30}$$

Parallel edges in G/e impose the same equality and are discarded.

The shift in degree and the Tate twist arise together from codimension one. We use the following framework input. If D is a smooth closed divisor in a smooth complex algebraic variety X, and $U = X \setminus D$, localization and purity give

$$R\Gamma(D, \mathbb{Q})(-1)[-2] \longrightarrow R\Gamma(X, \mathbb{Q}) \longrightarrow R\Gamma(U, \mathbb{Q}) \longrightarrow R\Gamma(D, \mathbb{Q})(-1)[-1]. \tag{31}$$

Here derived global sections retain the cochain complex and its maps; $[-2]$ places the divisor's contribution two cohomological degrees later. This is the localization triangle with support on D, together with the purity identification of that support complex. Its cohomology sequence is exact in mixed Hodge structures. Deligne supplies this structure and the strictness of its maps [33, §2.3, §3]; the corresponding arrangement argument is treated by Shapiro [34]. Our convention is that $\mathbb{Q}(-1)$ has type $(1,1)$ and weight 2.

Proof of Theorem 13.1. We first prove a slightly broader assertion: for the complement U of any finite arrangement of distinct affine hyperplanes, $H^k(U, \mathbb{Q})$ is pure of type (k,k) and vanishes above the ambient dimension. Induct on the number of hyperplanes, simultaneously in every ambient dimension. With no hyperplanes, affine space has only $H^0 = \mathbb{Q}(0)$.

Delete one hyperplane H. Let X be the complement of the remaining hyperplanes and $D = H \cap X$. Then D is a closed smooth divisor in X, and is itself a hyperplane-arrangement complement in H: omit empty intersections and identify repeated ones. Both X and D have fewer hyperplanes, so induction applies. The relevant part of (31) is

$$H^{k-2}(D)(-1) \xrightarrow{i_*} H^k(X) \longrightarrow H^k(U) \longrightarrow H^{k-1}(D)(-1) \xrightarrow{i_*} H^{k+1}(X).$$

The first Gysin map goes from weight $2k-2$ to weight $2k$; the last goes from weight $2k$ to weight $2k+2$. A morphism between pure Hodge structures of different weights is zero. Thus both maps vanish, giving

$$0 \longrightarrow H^k(X) \longrightarrow H^k(U) \longrightarrow H^{k-1}(D)(-1) \longrightarrow 0.$$

Both ends are pure of type (k,k). Exactness and strictness imply that the middle has the same type and weight. The sequence also proves the asserted vanishing, since D has one lower dimension. A rational Hodge structure of this type is a direct sum of $\mathbb{Q}(-k)$, proving the induction. The sequence is natural; a direct-sum decomposition requires a choice of splitting.

For a graph edge $e = uv$, take $H = \{z_u = z_v\}$. Then $X = \mathcal{C}_{G\setminus e}$, while identifying the two equal coordinates identifies D with $\mathcal{C}_{G/e}$. Hence the sequence above is (30), and

$$P_G(t) = P_{G\setminus e}(t) + tP_{G/e}(t). \tag{32}$$

Put $R_G(q) = q^n P_G(-1/q)$. Since contraction reduces the number of vertices by one, (32) becomes $R_G = R_{G\setminus e} - R_{G/e}$. The chromatic polynomial satisfies the same recurrence: colorings of $G \setminus e$ with equal colors at u, v are exactly colorings of G/e. For an edgeless graph both polynomials are q^n. Induction therefore gives $R_G = \chi_G$. Comparing coefficients proves the theorem. □

The numerical deletion–contraction identity has become an exact sequence of cohomology spaces. The decisive step was the vanishing forced by their weights. The Betti-number formula is also a consequence of the Orlik–Solomon description of arrangement cohomology [35, Theorem 5.2 and formula (1.4)]; the argument here exhibits the derived triangle that produces it.

Example 13.2 (The triangle's two cohomology degrees)**.** For K_3, quotient by simultaneous translation and put $a = z_1 - z_3$, $b = z_2 - z_3$. The forbidden lines are $a = 0$, $b = 0$, and $a = b$. Deleting the edge 12 gives $\mathcal{C}_{K_3\setminus 12} \cong \mathbb{A}^1 \times \mathbb{G}_m^2$; on its divisor $a = b$, the restriction is $\mathbb{A}^1 \times \mathbb{G}_m$. Consequently

$$P_{K_3}(t) = (1+t)^2 + t(1+t) = 1 + 3t + 2t^2,$$

so $H^1 = \mathbb{Q}(-1)^3$, $H^2 = \mathbb{Q}(-2)^2$, and $\chi_{K_3}(q) = q^3 - 3q^2 + 2q$. The residue of $(2\pi i)^{-1} d\log(a-b)$ is 1 on the divisor. Wedge it with $(2\pi i)^{-1} d\log a$ to obtain a class whose residue generates the divisor's first cohomology. Thus the extra degree-one and degree-two classes are visible in the residue maps themselves. For a tree, the edge differences and one root coordinate instead give $\mathcal{C}_G \cong \mathbb{A}^1 \times \mathbb{G}_m^{n-1}$, hence $P_G = (1+t)^{n-1}$.

13.2 Reduced coefficients and the motivic lift

Suppose now that G is connected and loopless, with $n \geq 2$. Quotient the coordinate space by simultaneous translation and let V be the resulting $(n-1)$-dimensional vector space. Its arrangement complement $\mathcal{U}_G$ satisfies $\mathcal{C}_G \cong \mathbb{A}^1 \times \mathcal{U}_G$. Let $\overline{\mathcal{U}}_G$ be the complement of the projectivized arrangement in $\mathbb{P}(V)$. Choose an edge form $f_e = z_u - z_v$. It never vanishes on $\mathcal{U}_G$, so the unique representative with $f_e = 1$ supplies an algebraic isomorphism

$$\mathcal{U}_G \cong \mathbb{G}_m \times \overline{\mathcal{U}}_G.$$

The projective complement is thus also an affine arrangement complement. Künneth gives $P_G(t) = (1+t)P_{\overline{\mathcal{U}}_G}(t)$, and the theorem gives

$$\frac{\chi_G(q)}{q(q-1)} = q^{n-2} P_{\overline{\mathcal{U}}_G}(-1/q).$$

The positive reduced characteristic coefficients μ_k are therefore

$$\mu_k = \dim_{\mathbb{Q}} H^k(\overline{\mathcal{U}}_G, \mathbb{Q}), \qquad H^k(\overline{\mathcal{U}}_G, \mathbb{Q}) \cong \mathbb{Q}(-k)^{\mu_k}. \tag{33}$$

For the square, $P_G = 1 + 4t + 6t^2 + 3t^3$ and $P_{\overline{U}_G} = 1 + 3t + 3t^2$. The three Betti numbers $(1, 3, 3)$ will reappear as the mixed intersection numbers in its wonderful compactification. The coefficient identity links these two realizations; Hodge–Riemann positivity in the compactification will then constrain the ranks of the open complement's Tate cohomology.

There is a motivic triangle behind the same deletion. For varieties over $\mathbb{Q}$, Voevodsky's Gysin theorem supplies

$$M(\mathcal{C}_G) \longrightarrow M(\mathcal{C}_{G\setminus e}) \longrightarrow M(\mathcal{C}_{G/e})(1)[2] \longrightarrow M(\mathcal{C}_G)[1] \tag{34}$$

in $DM_{\mathrm{gm}}(\mathbb{Q};\mathbb{Q})$ [36, Proposition 3.5.4]. Here a mixed Tate motive means an object in the thick triangulated subcategory generated by the Tate objects $\mathbb{Q}(j)$. Affine space has motive $\mathbb{Q}(0)$, and this subcategory is closed under twists, shifts, and cones. Induction on the hyperplanes in the triangle therefore proves that every coloring-space motive is mixed Tate. The triangle retains the maps that produced the recurrence, while the Hodge realization identifies each cohomology degree as a pure Tate space. This supplies both an object and a realization for the chromatic coefficients before we impose their intersection-theoretic inequality.

13.3 From edges to a graded intersection ring

In this section let G be connected and loopless, with at least two vertices. A graph with a loop has identically zero chromatic polynomial and can be handled separately. The graphic matroid $\mathcal{M}(G)$ retains which edge sets are forests. Its rank function is

$$\mathrm{rk}(S) = |V| - c(V, S),$$

where $c(V, S)$ counts components, including isolated vertices. This is also the rank of the incidence columns indexed by S. Thus a connected graph has matroid rank $r = |V| - 1$. A matroid abstracts this independence structure; it need not come from a graph or a configuration of vectors.

A *flat* is a subset F such that adding any outside element increases its rank. For a graphic matroid, F contains every graph edge whose endpoints are already connected in (V, F). The flats organize which dependencies have already been generated.

For a loopless matroid on a nonempty ground set E, introduce a degree-one variable x_F for each nonempty proper flat. Its real Chow ring is

$$A^*(\mathcal{M}) = \mathbb{R}[x_F]/(I + J),$$

where I is generated by $x_F x_H$ for incomparable flats, and J by

$$\sum_{F \ni i} x_F - \sum_{F \ni j} x_F, \qquad i, j \in E.$$

Put $d = r - 1$. The top degree is d, with a degree map normalized by

$$\deg(x_{F_1} \cdots x_{F_d}) = 1 \quad \text{for } F_1 \subsetneq \cdots \subsetneq F_d.$$

This ring turns flags of dependencies into products whose degrees are intersection numbers.

The arrangement of Section 13.2 realizes this graphic matroid: intersections of edge hyperplanes impose equality within the components of an edge set, so their codimensions equal the matroid ranks. A wonderful compactification of its projective complement produces the intersection ring above. This geometric realization is discussed in Section 5 of [23].

In this geometric realization, x_F is a divisor class, of codimension one, and multiplication records intersections of divisors. Chow degree one therefore corresponds to cohomological degree two. For a connected graph on N vertices, the top intersection degree is $N-2$. This grading belongs to the arrangement compactification, rather than the ordinary cohomology of the graph. The Adiprasito–Huh–Katz theorem establishes the corresponding intersection theory even for matroids without an arrangement realization.

Two distinguished classes are

$$\alpha = \sum_{F \ni i} x_F, \qquad \beta = \sum_{F \not\ni i} x_F.$$

Their classes are independent of i: the first assertion is a defining linear relation, and the second follows because $\alpha + \beta = \sum_F x_F$.

The positive classes needed for Hodge theory come from strictly submodular functions c on subsets, with $c_\varnothing = c_E = 0$:

$$c(S) + c(T) > c(S \cap T) + c(S \cup T) \quad \text{if } S, T \text{ are incomparable.}$$

They give $\ell(c) = \sum_F c(F) x_F$. For example $c(S) = |S||E \setminus S|$ works. The resulting ample cone is nonempty; a class in its closure is called nef. The classes α, β are nef. One can see this directly by averaging their representatives over $i \in E$: their coefficients become $|F|/|E|$ and $1 - |F|/|E|$. The corresponding functions on nonempty proper subsets, extended by zero at $\varnothing, E$, are weakly submodular. Adding $\varepsilon|S||E \setminus S|$ makes either function strictly submodular, and taking $\varepsilon \to 0$ exhibits the nef limits.

13.4 The deep theorem and its short numerical consequence

Theorem 13.3 (Adiprasito–Huh–Katz, cited input)**.** *For an ample class ℓ, multiplication by ℓ^{d-2j} gives*

$$A^j(\mathcal{M}) \longrightarrow A^{d-j}(\mathcal{M})$$

as an isomorphism for $0 \le j \le d/2$. The form

$$(-1)^j \deg(a\, b\, \ell^{d-2j})$$

is positive definite on the primitive subspace

$$\ker(\ell^{d-2j+1} : A^j \longrightarrow A^{d-j+1}).$$

These are hard Lefschetz and the Hodge–Riemann relations.

For $d \ge 2$, the useful sign in degree one is that $Q_\ell(a,b) = \deg(ab\ell^{d-2})$ is negative definite perpendicular to the positive direction ℓ.

Proposition 13.4 (The numerical inequality)**.** *For $d \ge 2$, any $a \in A^1$, and any nef $b \in A^1$,*

$$\deg(a^2 b^{d-2}) \deg(b^d) \le \deg(ab^{d-1})^2.$$

Proof. First take b ample. Write $a = tb + p$, where $\deg(pb^{d-1}) = 0$. Hard Lefschetz and Hodge–Riemann give $\deg(b^d) > 0$ and $Q_b(p,p) \le 0$. Consequently

$$Q_b(a,a)Q_b(b,b) = t^2 Q_b(b,b)^2 + Q_b(p,p)Q_b(b,b) \le Q_b(a,b)^2.$$

For nef b, choose an ample h and use the inequality for $b + \varepsilon h$, $\varepsilon > 0$. Letting $\varepsilon \to 0$ proves the assertion by continuity. □

This is a reversed Cauchy–Schwarz inequality, caused by the single positive direction of an intersection form. The connection to enumeration is another precise theorem: if

$$\overline{\chi}_{\mathcal{M}}(q) = \frac{\chi_{\mathcal{M}}(q)}{q-1} = \sum_{k=0}^{d} (-1)^k \mu_k q^{d-k},$$

then its positive integer coefficients satisfy

$$\mu_k = \deg(\alpha^{d-k}\beta^k). \tag{35}$$

This coefficient identity is Proposition 9.5 of [23]. Applying the numerical inequality with $a = \alpha, b = \beta$ gives

$$\mu_{d-2}\mu_d \le \mu_{d-1}^2.$$

All other indices follow by truncation. Truncating a rank-r matroid to rank $r-1$ keeps the independent sets of size at most $r-1$. Its characteristic polynomial has the same leading coefficients through rank $r-2$, and its reduced coefficients are $\mu_0, \dots, \mu_{d-1}$. To check this preservation, use the subset formula for $\chi_{\mathcal{M}}$ displayed below: a subset of rank at most $r-2$ retains its rank, while larger ranks only affect the last coefficient after truncation. Dividing by $q-1$ recovers each reduced coefficient successively from the leading coefficients. Applying induction to this matroid supplies the inequalities at the earlier indices. The ranks with $d \le 1$ have no interior inequality. Thus every reduced characteristic coefficient sequence is log-concave.

For a positive log-concave sequence $u_0, \dots, u_d$, put $v_i = u_i + u_{i-1}$, using zero outside the range, and write $r_i = u_i/u_{i-1}$. For $2 \le i \le d$, the ratio v_i/v_{i-1} is a weighted average of r_i and r_{i-1}, so lies between them. The first ratio is $r_1 + 1 \ge r_1$, and the last is $r_d/(1 + r_d) \le r_d$. Since the r_i decrease, all consecutive ratios of v decrease as well; $d = 0$ is immediate. Thus convolution with $(1, 1)$ preserves log-concavity. For the graphic arrangement, this convolution is precisely the Künneth decomposition of $\mathbb{G}_m \times \overline{\mathcal{U}}_G$.

13.5 Returning to graph colorings

Inclusion–exclusion over monochromatic edges gives

$$\chi_G(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{c(V,S)}.$$

The matroid characteristic polynomial is

$$\chi_{\mathcal{M}(G)}(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{r - \mathrm{rk}(S)}.$$

For connected G, $r - \mathrm{rk}(S) = c(V, S) - 1$. Hence

$$\chi_G(q) = q\, \chi_{\mathcal{M}(G)}(q).$$

Multiplication by q only appends a zero coefficient. The intersection inequality therefore proves log-concavity of the absolute chromatic coefficients. A one-vertex graph without loops has $\chi_G(q) = q$ and is immediate.

Example 13.5 (The square, calculated inside the Chow ring)**.** Label the four cycle edges $0, 1, 2, 3$. Every proper edge subset is independent, so the matroid is the rank-three uniform matroid $U_{3,4}$. Its nonempty proper flats are the singletons and pairs. Write their variables as x_i, x_{ij}. The degree normalization and relations give

$$\deg(x_i x_{ij}) = 1, \quad \deg(x_{ij}^2) = -1, \quad \deg(x_i^2) = -2.$$

To see the square identities, multiply $\alpha_i - \alpha_k = 0$ by x_{ij} for $k \notin \{i, j\}$, or multiply $\alpha_i - \alpha_j = 0$ by x_i. Incomparability kills the other products.

Now

$$\alpha = x_0 + x_{01} + x_{02} + x_{03}, \quad \beta = x_1 + x_2 + x_3 + x_{12} + x_{13} + x_{23}.$$

Expanding gives

$$\deg \alpha^2 = -2 - 3 + 6 = 1, \quad \deg \alpha\beta = 3, \quad \deg \beta^2 = -6 - 3 + 12 = 3.$$

Thus $\overline{\chi}(q) = q^2 - 3q + 3$, and the Hodge inequality reads $3^2 \geq 1 \cdot 3$. Multiplying by $q(q-1)$ yields

$$\chi_{C_4}(q) = q^4 - 4q^3 + 6q^2 - 3q.$$

The sequence $(1, 4, 6, 3)$ is log-concave. For comparison, K_4 has reduced matroid polynomial $(q-2)(q-3) = q^2 - 5q + 6$, corresponding to intersection numbers $(1, 5, 6)$, and chromatic coefficients $(1, 6, 11, 6)$.

The square makes the coefficient identity concrete. The same Hodge signature proves log-concavity for every graphic matroid and, more generally, for arbitrary matroids.

14 Reference layer: transport, attachments, and nearby cycles

The branch and gauge calculations above classify the stated matching data. The following results add global moduli and realization theorems. Tensor reconstruction, abstract blow-up descent, Picard representability and duality, Deligne full faithfulness, Andr'e comparison, and nearby-cycle theory are essential imported inputs here. The period-column splitting calculation does not establish those identifications by itself.

14.1 Tensor reconstruction of the transport category

Take $H = \mathrm{GL}(W)$. Edge transport defines a local system of vector spaces, and evaluating at a root gives a fiber functor ω. With tensor products retained,

$$\mathrm{Loc}_k(G) \simeq \mathrm{Rep}_k^{\mathrm{fd}}(F_r), \qquad \mathrm{Aut}^{\otimes}(\omega) \cong F_r^{\mathrm{alg}}.$$

The proalgebraic completion F_r^{alg} is the affine group scheme whose finite-dimensional algebraic representations recover the finite-dimensional k-linear representations of F_r. The category and root fiber functor therefore reconstruct the graph's transport symmetry [2, 3]. For a tree this category is equivalent to vector spaces, so the group is trivial. Motivic Galois theory uses the same construction for a specified neutral Tannakian category of motives and a chosen fiber functor.

A path and a star are both contractible, although their vertex degrees differ. Their local systems therefore have the same transport symmetry. Tree counts and currents depend on the incidence presentation and its edge coordinates; integral indices depend on the lattice. The curve construction also retains component curves and attachment points. We will see an attachment point force new motivic Galois symmetries even when the graph and its graded cohomology stay fixed. Each question determines which data we keep.

14.2 The normalization triangle

Let C be a connected projective nodal curve over an algebraically closed field k, with normalization

$$\nu : C^\nu = \coprod_{v \in V(\Gamma)} C_v \longrightarrow C.$$

The dual graph has a vertex for each smooth normalized component and an edge for each node. Self-nodes give loops; repeated attachments give parallel edges. Write $n = |V|$, $m = |E|$ and $r = m - n + 1$, and choose branch points s_e, t_e according to an orientation. Every finite connected graph occurs: choose distinct points for its half-edges on smooth projective components and identify the prescribed pairs.

Theorem 14.1 (Normalization is a derived descent diagram)**.** *Over $\mathbb{C}$ there is a distinguished triangle of mixed Hodge complexes*

$$R\Gamma(C,\mathbb{Q}) \longrightarrow \bigoplus_v R\Gamma(C_v,\mathbb{Q}) \xrightarrow{t^*-s^*} \mathbb{Q}^E \longrightarrow R\Gamma(C,\mathbb{Q})[1]. \tag{36}$$

In particular, $R\Gamma(C,\mathbb{Q})$ is the homotopy fiber of the branch-value map. Over an algebraically closed field of characteristic zero, its motivic counterpart is

$$\mathbb{Q}^E \xrightarrow{t_*-s_*} M(C^\nu) \longrightarrow M(C) \longrightarrow \mathbb{Q}^E[1] \quad \text{in } DM_{\mathrm{gm}}(k;\mathbb{Q}). \tag{37}$$

Here M denotes the geometric motive and $\mathbb{Q} = M(\operatorname{Spec} k)$.

Proof. Let S be the nodes and $\widetilde{S} = \nu^{-1}(S)$ their two branches. The square with upper row $\widetilde{S} \to C^\nu$ and lower row $S \to C$ is an abstract blow-up square: ν is proper and is an isomorphism away from S. Motivic descent gives

$$M(\widetilde{S}) \longrightarrow M(C^\nu) \oplus M(S) \longrightarrow M(C) \longrightarrow M(\widetilde{S})[1].$$

There are two copies of $\mathbb{Q}$ over each node. One maps isomorphically to the node's copy in $M(S)$; cancelling it leaves the difference of the two point maps. This gives (37). The framework input is the abstract blow-up triangle [36, Proposition 4.1.3].

Applying the contravariant Betti cochain realization gives (36). Directly, the same triangle comes from $0 \to \mathbb{Q}_C \to \nu_*\mathbb{Q}_{C^\nu} \to \bigoplus_e \mathbb{Q}_e \to 0$: a locally constant function descends exactly when its two branch values agree. Mixed Hodge realization equips this descent triangle with its filtrations. The point maps, rather than only their dimensions, remain part of the construction. □

Taking degree one gives the extension

$$0 \longrightarrow H^1(\Gamma,\mathbb{Q}) \longrightarrow H^1(C,\mathbb{Q}) \longrightarrow H^1(C^\nu,\mathbb{Q}) \longrightarrow 0. \tag{38}$$

Its outer terms are pure of weights 0 and 1, respectively. Thus $\mathrm{Gr}_0^W H^1(C,\mathbb{Q}) = H^1(\Gamma,\mathbb{Q})$, while $\mathrm{Gr}_1^W H^1(C,\mathbb{Q}) = \bigoplus_v H^1(C_v,\mathbb{Q})$. Taking the weight-zero part leaves the incidence complex $\mathbb{Q}^V \xrightarrow{\delta} \mathbb{Q}^E$, with $(\delta a)_e = a_{t(e)} - a_{s(e)}$. The full triangle retains the attachment maps that this graded piece discards.

The coherent version is the homotopy fiber of $R\Gamma(C^\nu,\mathcal{O}) \to k^E$, obtained from the branch-difference sequence for regular functions. Its degree-one sequence gives

$$p_a(C) = \sum_v g(C_v) + r.$$

The genus formula and the weight-zero formula are two cohomological consequences of the same normalization diagram.

14.3 Multiplicative descent and derived Cartier duality

For line bundles, branch values are replaced by identifications of one-dimensional fibers. We use gluing maps from the fiber at t_e to the fiber at s_e, and branch ratios $h \mapsto h(s_e)/h(t_e)$. The exact sequence of *étale* sheaves $1 \to \mathbb{G}_{m,C} \to \nu_*\mathbb{G}_{m,C^\nu} \to \bigoplus_e i_{e*}\mathbb{G}_{m,k} \to 1$ uses this ratio map. Finite pushforward is exact on abelian étale sheaves [37, Proposition 59.55.2]; hence it yields a triangle

$$R\Gamma(C,\mathbb{G}_m) \longrightarrow R\Gamma(C^\nu,\mathbb{G}_m) \xrightarrow{(s_e^*/t_e^*)_e} (k^\times)^E \longrightarrow R\Gamma(C,\mathbb{G}_m)[1].$$

Its degree-zero and degree-one terms compute units and Picard groups. Descent in families gives the corresponding Picard-functor sequence. With $A = \mathrm{Pic}^0(C^\nu)$ and $L = H_1(\Gamma,\mathbb{Z})$, it reads

$$1 \longrightarrow T_\Gamma \longrightarrow J(C) \xrightarrow{\nu^*} A \longrightarrow 0, \qquad T_\Gamma = \mathrm{Hom}(L,\mathbb{G}_m). \tag{39}$$

Indeed, component rescalings act on gluing parameters by $\lambda_e \mapsto a_{s(e)}\lambda_e a_{t(e)}^{-1}$. Characters of their quotient are the integral edge vectors annihilating the incidence map: precisely L. Here $J(C) = \mathrm{Pic}^0(C)$ consists of line bundles of degree zero on every component.

An extension of A by this torus is classified by a homomorphism $c_C : L \to A^\vee$: push out along each torus character and use $\mathrm{Ext}^1(A,\mathbb{G}_m) = A^\vee$. The product principal polarization $\lambda_A : A \to A^\vee$ gives the explicit formula

$$c_C(z) = \lambda_A\left[\sum_e z_e(t_e - s_e)\right], \qquad z \in L. \tag{40}$$

The cycle condition makes the divisor have degree zero on each component.

Theorem 14.2 (The gluing extension is the derived dual of its attachment map)**.** *Put $M_C = [L \xrightarrow{c_C} A^\vee]$, with L in degree -1. In the derived category of abelian fppf sheaves over k,*

$$J(C) \cong \mathcal{H}^1 R\mathcal{H}om(M_C,\mathbb{G}_m). \tag{41}$$

Equivalently, this Deligne 1-motive is Cartier dual to $[0 \to J(C)]$. Its homological realizations recover the dual of the curve's first cohomology; over $\mathbb{C}$, $T_H(M_C)\otimes\mathbb{Q} \cong H^1(C,\mathbb{Q})^\vee$.

Proof. Choose rigidified universal Poincaré bundles on the normalized components. Over a point of A, the gluing parameter at e is a nonzero element of $\mathcal{P}_{s_e}\otimes\mathcal{P}_{t_e}^{-1}$. Abel–Jacobi pullback is minus the inverse of the usual principal polarization [15, Lemma 6.9 and Summary 6.11]. Pushing out by the character z tensors these lines with powers z_e; the cycle condition cancels the rigidifying basepoints. Its class is therefore the image under λ_A of the divisor in (40). This proves the classifying formula, including its attachment dependence [14, §3.0–3.1.1].

The complex M_C lies in a triangle $A^\vee \to M_C \to L[1] \to A^\vee[1]$. Apply $R\mathcal{H}om(-,\mathbb{G}_m)$. Since L is locally free, $R\mathcal{H}om(L,\mathbb{G}_m) = T_\Gamma$ in degree zero. Also $\mathcal{H}om(A^\vee,\mathbb{G}_m) = 0$ and $\mathcal{E}xt^1(A^\vee,\mathbb{G}_m) = A$. The degree-one sequence is therefore $0 \to T_\Gamma \to \mathcal{E}xt^1(M_C,\mathbb{G}_m) \to A \to 0$. Its points are extensions of $A^\vee$ by $\mathbb{G}_m$, together with a trivialization after pullback along c_C. Poincaré duality identifies its classifying map with c_C, so this is (39). Representability and the compatible realizations are Deligne's duality construction [16, §10.2.10–10.2.13, §10.3]. □

The Hodge realization is concrete. If $\exp : \mathrm{Lie}(A^\vee) \to A^\vee(\mathbb{C})$ is the exponential map, its integral lattice is

$$T_\mathbb{Z}(M_C) = \{(z,u) : z \in L,\ u \in \mathrm{Lie}(A^\vee), \quad \exp(u) = c_C(z)\}.$$

It fits into $0 \to H_1(A^\vee, \mathbb{Z}) \to T_{\mathbb{Z}}(M_C) \to L \to 0$: a lift of a graph circuit includes paths between its attachment points, and changing those paths adds a component cycle. Dualizing recovers (38), with weights 0 and 1. For the ℓ-adic realization one uses derived torsion,

$$M_C[\ell^a] = \mathcal{H}^{-1}(M_C \otimes^L \mathbb{Z}/\ell^a), \qquad T_\ell(M_C) = \varprojlim_a M_C[\ell^a].$$

Over a number field K, the resulting comparison $T_\ell(M_C) \otimes \mathbb{Q}_\ell \cong H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{Q}_\ell)^\vee$ is Galois compatible. The de Rham realization uses the universal vector extension and is compared over $\mathbb{C}$ with $T_{\mathbb{Z}}(M_C) \otimes \mathbb{C}$. When Galois permutes components or exchanges branches, it acts on L; the torus with this character lattice can be nonsplit.

Example 14.3 (Two elliptic components, two nodes)**.** Glue $Q \in E_1$ to $R \in E_2$, and $P \in E_1$ to $S \in E_2$, with distinct attachment points on each elliptic curve. Orient both edges from E_1 to E_2. For $z = e_1 - e_2$, under the principal polarizations,

$$c_C(z) = (P - Q, R - S), \qquad M_C = [\mathbb{Z} \xrightarrow{1 \mapsto (P-Q, R-S)} E_1 \times E_2].$$

Thus $p_a(C) = 3$, $\dim H^1(C, \mathbb{Q}) = 5$, and $1 \to \mathbb{G}_m \to J(C) \to E_1 \times E_2 \to 0$ is the extension represented by this point. Moving P changes its class with the two end groups fixed. A nontorsion attachment class remains nontrivial after passing to 1-motives up to isogeny. The graph records its rank; the derived dual records the point that defines the extension.

For a tree, $L = 0$: the Jacobian is A, the first cohomology is pure of weight 1, and the genus is the sum of the component genera. Every node is separating. At the other extreme, a rational nodal cubic has $A = 0$, $M_C = [\mathbb{Z} \to 0]$, $J(C) = \mathbb{G}_m$ and $H^1(C, \mathbb{Q}) = \mathbb{Q}(0)$. A separating node contributes no cycle divisor. In particular, gluing two fixed nonisomorphic curves of genus at least two at varying point pairs keeps their product Jacobian fixed; their finite automorphism groups leave many different nodal curves with that same Jacobian.

14.4 The attachment point changes the motivic symmetry group

The attachment map gives a concrete object to reconstruct by the tensor method of Section 8. Consider an elliptic curve E with origin O and identify O with a distinct point Q to form a nodal curve C_Q. Its dual graph has one vertex and one loop. The normalization formulas give

$$p_a(C_Q) = 2, \qquad \mathrm{Gr}^W_0 H^1(C_Q) = \mathbb{Q}(0), \qquad \mathrm{Gr}^W_1 H^1(C_Q) = H^1(E).$$

These dimensions are independent of Q. The attachment map retains the difference between a torsion point and a point of infinite order.

To make the motivic symmetry group precise, work over an algebraically closed field $k \subset \mathbb{C}$, with E and Q defined over k. Let

$$M_Q = [\mathbb{Z} \xrightarrow{1 \mapsto \lambda_E[Q-O]} E^\vee], \qquad H_Q = T_H(M_Q) \otimes \mathbb{Q}, \qquad V = H_1(E^\vee, \mathbb{Q}).$$

Here λ_E is the principal polarization, with the gluing convention fixed above. Let $\mathcal{M}_Q = \langle M_Q \rangle^\otimes$ be the full Tannakian subcategory of rational Nori motives generated by M_Q: we allow tensor products, duals, finite direct sums and subquotients. Its Betti fiber functor ω_B defines

$$G_Q = \mathrm{Aut}^\otimes(\omega_B|_{\mathcal{M}_Q}).$$

André's comparison theorem identifies this category, through Hodge realization, with the tensor category generated by H_Q inside rational mixed Hodge structures. Thus G_Q is also the Mumford–Tate group of H_Q [39, Theorem 1.2.1]. This is the deep input that gives the following Hodge calculation its motivic meaning.

Theorem 14.4 (Torsion and the unipotent symmetry of gluing)**.** *Put $G_E = \mathrm{MT}(V)$. There is an exact sequence of algebraic $\mathbb{Q}$-groups*

$$1 \longrightarrow U_Q \longrightarrow G_Q \longrightarrow G_E \longrightarrow 1, \qquad U_Q = \begin{cases} 0, & Q \text{ is torsion}, \\ \mathbf{V}, & Q \text{ has infinite order}, \end{cases}$$

where $\mathbf{V}$ is the additive vector group of the two-dimensional space V. The group U_Q is the unipotent radical of G_Q. If E has no complex multiplication, then

$$G_Q \cong \begin{cases} \mathrm{GL}_2, & Q \text{ is torsion}, \\ \mathbb{G}_a^2 \rtimes \mathrm{GL}_2, & Q \text{ has infinite order}, \end{cases} \qquad \dim G_Q = \begin{cases} 4, & Q \text{ is torsion}, \\ 6, & Q \text{ has infinite order}. \end{cases}$$

The semidirect-product description uses a choice of splitting on the underlying rational vector space; the exact sequence and its radical are intrinsic.

Proof. The realization lattice from the preceding subsection gives an exact sequence of mixed Hodge structures

$$0 \longrightarrow V \longrightarrow H_Q \longrightarrow \mathbb{Q}(0) \longrightarrow 0.$$

First, this sequence splits over $\mathbb{Q}$ exactly when Q is torsion. Deligne's full faithfulness for 1-motives transfers a Hodge section to a section of $[\mathbb{Z} \to E^\vee] \to [\mathbb{Z} \to 0]$ up to isogeny [16, §10.1]. After clearing denominators, such a section has a nonzero integer m as its lattice map and zero as its map from the trivial group to $E^\vee$. Commutativity requires $m\lambda_E[Q - O] = 0$. Conversely, if this relation holds, the lattice map m, divided by m in the category up to isogeny, supplies a section.

Tannakian reconstruction identifies $\mathcal{M}_Q$ with representations of G_Q, and the subcategory generated by V gives the surjection $G_Q \to G_E$. Choose a rational vector-space splitting $H_Q \cong V \oplus \mathbb{Q}$. Every group element then has the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \qquad a \in G_E, \quad b \in V.$$

Consequently the kernel K is an algebraic subgroup of the additive vector group $\mathbf{V}$. In characteristic zero such a subgroup is a linear subspace. Conjugation makes it a G_E-submodule of V.

The pure polarizable Hodge structure V is simple. Indeed, its Hodge types are $(-1, 0)$ and $(0, -1)$, so complex conjugation rules out a one-dimensional rational Hodge substructure. Hence V is an irreducible G_E-module and K is either zero or all of $\mathbf{V}$. Pure polarizable Hodge structures form a semisimple tensor category; therefore G_E is reductive [40, §6, Corollary 6.5]. The kernel is connected and unipotent, and every normal unipotent subgroup of G_Q maps trivially to this reductive quotient. Thus K is the unipotent radical.

If $K = 0$, then $G_Q \cong G_E$ is reductive. Its representations are semisimple, so the displayed extension of $\mathbb{Q}(0)$ by V splits. For a point of infinite order this is impossible; hence $K = \mathbf{V}$. For a torsion point the splitting gives $\langle H_Q \rangle^\otimes = \langle V \rangle^\otimes$, and $K = 0$.

Finally, the usual pure elliptic calculation gives $G_E = \mathrm{GL}(V)$ when E has no complex multiplication [41, Example 5.11]. In the nonsplit case the kernel contains every translation and the

quotient contains every $a \in \mathrm{GL}(V)$. Thus the block matrices above form the entire group, with group law $(a,b)(a',b') = (aa', ab'+b)$. In the split case a Hodge section gives the block-diagonal copy of $\mathrm{GL}(V)$. This proves both dimensions. □

Example 14.5 (The same loop, with symmetry dimensions four and six)**.** Take

$$E:\ y^2 = x^3 - x + 1, \qquad P = (0,1).$$

The tangent slope at P is $-1/2$, so $2P = (1/4, -7/8)$. Rational torsion points on this integral short Weierstrass equation have integral coordinates by the Lutz–Nagell theorem [42, II, Theorem 5.1]. Thus P has infinite order. Moreover,

$$j(E) = 1728\,\frac{4(-1)^3}{4(-1)^3 + 27} = -\frac{6912}{23}.$$

A CM elliptic curve has an algebraic-integer j-invariant [43, Corollaries 12.15–12.16]; this rational noninteger shows that E has no complex multiplication.

Let α be a root of $x^3 - x + 1$ and put $T = (\alpha, 0)$. Over $\overline{\mathbb{Q}} \subset \mathbb{C}$, the point T is a nonzero point of order two. The curves $C_T = E/(O \sim T)$ and $C_P = E/(O \sim P)$ have the same normalization, the same one-loop dual graph, arithmetic genus two, and weight-graded first cohomology of dimensions $(1,2)$. Their associated motivic groups nevertheless have dimensions

$$\dim G_T = 4, \qquad \dim G_P = 6.$$

On H_P, component-homology translations act transitively on the lifts of the graph circuit; equivalently, $H_P^{U_P} = V$. Passing to the weight-graded object gives the same pure symmetry group. Keeping the attachment extension forces, in the second case, every translation of the elliptic homology space.

The graph supplied one circuit. The attachment point determined whether the extension split up to isogeny, and tensor-compatible symmetry turned that distinction into a computed unipotent radical. Nearby cycles add a different structure to the same circuit: its monodromy pairing.

14.5 Nearby cycles put the graph at both ends of the weight filtration

Let $f : \mathcal{C} \to \Delta$ be a projective semistable smoothing over a complex disk, with smooth total space, reduced nodal central fiber C, and smooth fibers away from 0. Locally at a node, f has equation $xy = t$. Nearby cycles form a complex on C whose first hypercohomology is the limiting cohomology $H^1_{\lim}$. Its Hodge filtration comes from the relative logarithmic de Rham complex $\Omega^\bullet_{\mathcal{C}/\Delta}(\log C)|_C$.

Theorem 14.6 (Weights and invariant cycles for a semistable curve)**.** *Put $H = H^1_{\lim}$ and $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$. Then*

$$\mathrm{Gr}^W_0 H = L^\vee_{\mathbb{Q}}, \qquad \mathrm{Gr}^W_1 H = H^1(C^\nu, \mathbb{Q}), \qquad \mathrm{Gr}^W_2 H = L_{\mathbb{Q}}(-1).$$

Specialization identifies $H^1(C,\mathbb{Q})$ with $W_1 H = \ker N$, where $N : H \to H(-1)$ is the logarithm of monodromy. The induced map $\mathrm{Gr}^W_2 H \to \mathrm{Gr}^W_0 H(-1)$ is the nondegenerate cycle pairing $q(z,z') = \sum_e z_e z'_e$. Consequently $N^2 = 0$, $\operatorname{rank} N = r$, and $N = 0$ exactly when Γ is a tree.

Proof. The framework input is the nearby-cycle mixed Hodge complex and its weight spectral sequence. For curves its relevant rows are

$q\backslash p$	-1	0	1
2	$\mathbb{Q}^E(-1)$	$\xrightarrow{\partial} \mathbb{Q}^V(-1)$	0
1	0	$H^1(C^\nu,\mathbb{Q})$	0
0	0	$\mathbb{Q}^V \xrightarrow{\delta}$	$\mathbb{Q}^E$.

Restriction gives δ; the signed Gysin maps from node points to components give ∂. The three terms of total degree 1 on the second page are therefore $\ker\partial$, $H^1(C^\nu)$ and $\operatorname{coker}\delta$, with the displayed twists. Their weights are respectively $2, 1, 0$. Higher differentials vanish because they would map between pure Hodge structures of different weights. This computes the three graded pieces. Steenbrink's complex and its curve case are described in [38, §1, §4].

The nearby–vanishing-cycle triangle is $\mathbb{Q}_C \to R\psi_f\mathbb{Q} \to \bigoplus_e \mathbb{Q}_e(-1)[-1] \to \mathbb{Q}_C[1]$. Its cohomology boundary is the same ∂, so it gives $0 \to H^1(C) \xrightarrow{\mathrm{sp}} H \to L_{\mathbb{Q}}(-1) \to 0$. Picard–Lefschetz identifies N on the two graph pieces with the map $\ker\partial \hookrightarrow \mathbb{Q}^E \to \operatorname{coker}\delta$ induced by the edge pairing; the normalization part is killed [17, §6.1, (25)–(29)]. The edge pairing restricts to a positive definite form on the real cycle space. Hence its map $L_{\mathbb{Q}} \to L_{\mathbb{Q}}^\vee$ is an isomorphism. It follows that $\ker N = \mathrm{sp}(H^1(C)) = W_1H$ and that N has rank r and square zero. This also proves the degree-one invariant-cycle statement and the asserted monodromy-weight isomorphism. □

In the two-elliptic-component example, $\dim H = 6$ and its weight dimensions are $1, 4, 1$. The central fiber contributes the five invariant dimensions; one further nearby class maps under N to the graph class. On $z = e_1 - e_2$, the pairing is $q(z, z) = 2$.

Corollary 14.7 (The graph polynomial is the determinant of a monodromy pairing)**.** *For a smoothing with local node equations $xy = t^{\ell_e}$, where $\ell_e \geq 1$, the graph pairing is $q_\ell(z, z') = \sum_e \ell_e z_e z'_e$. In an integral cycle basis with edge matrix Z, its determinant is*

$$\det(q_\ell) = \det\big(Z\,\mathrm{diag}(\ell_e)Z^{\mathsf{T}}\big) = \Psi_\Gamma(\ell).$$

Proof. A loop in the smoothing disk circles the node's deformation parameter ℓ_e times, so $N = \sum_e \ell_e N_e$. The Picard–Lefschetz edge maps in the preceding proof give q_ℓ. Restricting the diagonal edge form to the cycle lattice gives the displayed matrix. The total space may be singular for $\ell_e > 1$; the claim concerns its nearby monodromy. □

This is a geometric route to the graph polynomial of the next section: its variables are the weights of a form on the cycle lattice.

15 Reference layer: motivic localization and global period theorems

The coordinate elimination in Section 10.1 proves a partition of varieties. Its lift to motives uses the following compact-support localization theorem; this extra structure is not obtained from point counts.

Theorem 15.1 (Motivic deletion–contraction localization)**.** *Over $\mathbb{Q}$, in $DM_{\mathrm{gm}}(\mathbb{Q};\mathbb{Q})$, there is a distinguished triangle*

$$M^c(Z_0)(1)[2] \longrightarrow M^c(U_G) \longrightarrow M^c(O) \otimes M^c(\mathbb{G}_m) \longrightarrow M^c(Z_0)(1)[3]. \tag{42}$$

Here M^c denotes the motive with compact support. If $M^c(O)$ and $M^c(Z_0)$ are mixed Tate, then so is $M^c(U_G)$.

Proof. Projection to the variables other than α_e lands in B. Over O, the coordinate $u = \alpha_e + H/F$ gives an isomorphism with $O \times \mathbb{G}_m$. Over Z_0, every α_e is allowed, giving $Z_0 \times \mathbb{A}^1$. The latter is a

closed subvariety of U_G. The compact-support localization triangle for this closed subvariety, and $M^c(X \times \mathbb{A}^1) = M^c(X)(1)[2]$, give (42) [36, Propositions 4.1.5–4.1.7, Corollary 4.1.8]. Localization of $\{0\} \subset \mathbb{A}^1$ makes $M^c(\mathbb{G}_m)$ mixed Tate. The mixed Tate subcategory is the thick tensor subcategory generated by the Tate objects $\mathbb{Q}(a)$, so the assertion follows from the triangle. □

The symbols O and Z_0 retain their definitions from that coordinate calculation. The triangle preserves connecting maps that the scissor identity discards.

15.1 Convergence and the projective period

Graph hypersurfaces also support period integrals. Suppose $r \geq 1$, $m = 2r$, and every nonempty proper edge subgraph γ satisfies $|E(\gamma)| > 2b_1(\gamma)$. These primitive logarithmic-divergence conditions give a convergent projective period

$$P(G) = \int_\sigma \frac{\Omega}{\Psi_G^2}, \qquad \Omega = \sum_{i=1}^{m} (-1)^{i-1} \alpha_i \, d\alpha_1 \wedge \cdots \wedge \widehat{d\alpha_i} \wedge \cdots \wedge d\alpha_m,$$

where σ is the positive projective simplex and its orientation makes the integrand positive. Equivalently, fix $\alpha_m = 1$ and integrate over the positive orthant in the other variables. These conditions and the relative-cohomology construction are given in [18, §5, §7].

15.2 The period is a pairing attached to a relative motive

The simplex boundary can meet the graph hypersurface. The blowups in [18, §7] produce a smooth space P, the strict transform Y of the projective hypersurface, and a boundary divisor D. Put $V = P \setminus Y$ and $B_D = D \setminus (D \cap Y)$. The relative homological motive and its Betti cochain realization are

$$M(V, B_D) = \mathrm{Cone}\big(M(B_D) \to M(V)\big), \qquad K_B(V, B_D) = \mathrm{Cone}\big(R\Gamma(V, \mathbb{Q}) \to R\Gamma(B_D, \mathbb{Q})\big)[-1].$$

The shift comes from dualizing relative chains. In degree $m-1$ the algebraic de Rham realization gives the relative cohomology used by the integral. Comparison over $\mathbb{C}$ carries the pulled-back class $[\Omega/\Psi_G^2]_{\mathrm{dR}}$ to Betti cohomology; the transformed simplex is a class $[\widetilde{\sigma}]_B$ in relative Betti homology. Their evaluation is

$$P(G) = \left\langle \mathrm{comp}_{\mathrm{dR},B}[\Omega/\Psi_G^2]_{\mathrm{dR}}, [\widetilde{\sigma}]_B \right\rangle.$$

The relative cone keeps the boundary condition as part of the object. Blow-up and localization triangles carry this object through the change of geometry: for a smooth center Z of codimension c, $M(\mathrm{Bl}_Z P) \cong M(P) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i]$ [36, Proposition 3.5.3]. The graph hypersurface and its intersections with the boundary then enter through localization. This relative object carries the period and its boundary data. The evaluations and arithmetic results below require further theorems, which we cite.

For the wheel W_r with $r \geq 3$ spokes, $m = 2r$ and this normalization gives the established evaluation

$$P(W_r) = \binom{2r-2}{r-1} \zeta(2r-3).$$

The three- and four-spoke wheels give $6\zeta(3)$ and $20\zeta(5)$. This normalization is Definition 61 and Theorem 62 with $k = 1$ of [20]; those Feynman periods are distinct from the paper's canonical trace-form integrals.

Brown's theorem applies to an infinite family. For an ordering of the edges, count the vertices incident both to an initial segment and to the remaining edges. The vertex-width is the minimum, over all orderings, of the largest such count. Brown proves that graphs of vertex-width at most 3 admit a linear-reduction structure: the graph motives in his construction are mixed Tate, and convergent primitive periods are rational linear combinations of multiple zeta values. The proof organizes successive integrations through linear fibrations and the moduli spaces $M_{0,N}$, where monodromy can be controlled uniformly [21, Introduction, Theorems 1–3].

Brown and Schnetz construct an eight-loop ϕ^4 graph whose hypersurface point counts detect a modular form arising from a singular K3 surface [22]. This establishes non-Tate graph-hypersurface arithmetic. Whether the numerical period belongs to the algebra of multiple zeta values also involves the part of cohomology selected by the integral; the point counts alone do not settle that question.

The integral is a further question, not a consequence of knowing the cycle rank or the tree count. Writing its denominator as a determinant allows one to test elimination orders and to identify a relative cohomology class, but evaluating the period requires additional theorems. Brown's linear-reduction condition supplies an effective route in one family; the K3 example shows a different arithmetic possibility. The useful representation makes both the available method and its limits visible.

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